Geometric properties of generalized sheaves of conformal blocks

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$/\mathbb{C}$

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What are we going to see today?

rep. theor.

V vertex op. algebra

$M_\bullet$ V-modules

Describe properties of

the sheaf of coinvariants

$$\mathbb{V}_g(V; M_1 \dots M_n)$$
$$\downarrow$$
$$\overline{\mathcal{M}}_{g,n}$$

their dual are called conformal blocks, $\mathbb{V}^\dagger$

moduli space of n-marked stable curves of genus g

→ get more info about $\overline{\mathcal{M}}_{g,n}$

→ $\mathbb{V}^\dagger$ appear also in the study of Bun_G

① 

② $\mathbb{V}_g(M_\bullet)$ collection of sheaves

g, n varying

Back to the origins: coinvariants from Lie algebras

Geometry

$C, P_1 \text{---} P_n$

nodal, P_i smooth

need P on each comp. of C

$\longrightarrow$ vector space

varying (C, P) in $\overline{M}_{g,n}$

Rep theory

$\mathfrak{g}$ simple Lie algebra $/\mathbb{C}$, $\ell \in \mathbb{Z}_{\geq 0}$ level

$W_1 \text{---} W_n$ irr. repr of $\mathfrak{g}$ at $\underline{\text{level } \ell}$

$\mathfrak{g} = \mathfrak{sl}_2$

$\dim W \leq \ell + 1$

$$\mathbb{V}\left((C, P_\bullet)(\mathfrak{g}, \ell, W_\bullet)\right) = \left[\frac{\text{quotient of thickening of } \otimes W_i}{(\mathfrak{g} \otimes \mathcal{O}_{C \setminus P_\bullet})(\ \ )}\right.$$

this defines a sheaf

around P_i

$D^{\times}_{P_i}$

Classical Results

TUY '89 Tsuchimoto

DREAM: This holds replacing $\mathfrak{g}, \ell W.$ with (V.M.)

When $(C, P.)$ varies in $\overline{\mathcal{M}}_{g,n}$ the spaces? $\mathbb{V}((C,P.)(\mathfrak{g},\ell,W.))$ fit together to define a VECTOR BUNDLE of finite rank? over $\overline{\mathcal{M}}_{g,n}$ denoted $\mathbb{V}_g(\mathfrak{g},\ell,W.)$. Its dual, $\mathbb{V}_g(\mathfrak{g},\ell,W.)^{\dagger}$ is called the sheaf of conformal blocks.

Properties

Tsuchimoto
- There exists a proj. flat connection on $\mathbb{V}_g(\mathfrak{g},\ell,W)$ on $M_{g,n}$

Faltings, F. MOPPZ
- Rank & Chern char. are explicitly computed
  - Rank ↑ verlinde formula
  - Chern char. using Cohom. field Theory

? Beauville Laszlo
- $\boxed{\circ}$ $\mathbb{V}((C,P)(\mathfrak{g},\ell,\mathbb{C}))^{\dagger} \cong H^0(\mathrm{Bun}_{G,C}, \Theta^{\ell})$ $\rightsquigarrow$ comp. of $\mathrm{Bun}_G \to \overline{\mathcal{M}}_{g,n}$

G = simply conn. simple, $\mathrm{Lie}\, G = \mathfrak{g}$

$\mathrm{Pic}\,\mathrm{Bun}_G = \mathbb{Z}\Theta$

U.F.
- on $\overline{\mathcal{M}}_{0,n}$, the coinvariants $\mathbb{V}_0(\mathfrak{g},\ell,W.)$ are quotient of $\otimes W.$

Generalizations of these sheaves

	Geometry	Representation Theory
Classical	$C, p_1 \cdots p_n$	$\underbrace{\mathfrak{g}, \ell,}_{V \text{ vertex alg}}$ $\overbrace{W_1 \cdots W_n}^{M_1 \cdots M_n \in V\text{-mod.}}$
V.O.A.		

$x_i \in \mathfrak{g} \quad r \geq 0 \quad n_1 \geq \cdots \geq n_r$

$$\mathfrak{g}, \ell \rightsquigarrow V_\ell(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}_\ell} \mathbb{C} v_0 = \langle X_1 t^{-n_1} \otimes \cdots \otimes X_r t^{-n_r} \otimes v_0 \rangle_{\mathbb{C}}$$

$$= \boxed{\mathbb{C} v_0} \oplus (\mathfrak{g} t^{-1} \otimes v_0) \oplus \mathfrak{g} t^{-1} \otimes \mathfrak{g} t^{-1} v_0 \oplus \mathfrak{g} t^{-2} v_0$$

graded vector space

$\boxed{\omega}$ in $V_\ell(\mathfrak{g})_2$

$c_V \in \mathbb{Q}$ central charge

thicker Lie algebra

$$\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C} 1_\ell = \hat{\mathfrak{g}}_\ell$$

$$(Xt^{-1} v_0, n) \longmapsto Xt^n \cdot \_\_$$

oper of degree $\boxed{-n}$

What is a Vertex Operator Algebra?

$V = \bigoplus_{i \geq 0} V_i$ $\quad$ $\dim V_i < \infty$

$Y(-,z): V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]]$

$A \longmapsto \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$

$\mathbb{1} \cdot \mathbb{C} = V_0$

$Y(\mathbb{1}) = \mathrm{Id}_V$

$\boxed{\begin{array}{c}\omega \in V_2 \\ \text{conf.} \\ \text{vector}\end{array}}$

PLAY ROLE in background

$\{\omega_{(n)}, \boxed{c_V} \cdot \mathrm{Id}_V\} \cong$ Virasoro Lie alg.

$\uparrow$ central charge of V

$\left(V = V_\ell(\mathfrak{g}) \qquad c_V = \dfrac{\ell \cdot \dim \mathfrak{g}}{\ell + \mathfrak{g}^*} \right)$

What is a V-module?

V simple

$M = \bigoplus M_i$ $\quad$ $\dim M_i < \infty$

$Y^M(-,z): V \to \mathrm{End}(M)[[z, z^{-1}]]$

$A^M_{(n)}$ has degree $\deg A - n - 1$

$\omega_{(1)}$ acts on degree d as $(d + \overbrace{a_M}^{\in \mathbb{C}})\,\mathrm{Ident}_{M_d}$

conformal weight

Coinvariants associated with (C,P) and (V,M)

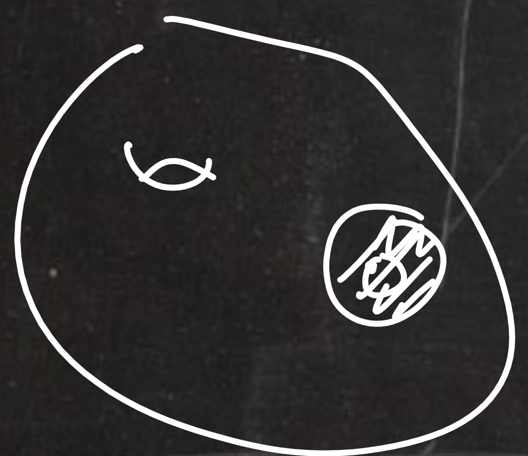
$$\mathbb{V}((C,P)(V,M)) = \frac{M}{[\text{Lie alg encoding } C\setminus P \text{ action on } M]} = \frac{M}{\mathcal{L}_{C\setminus P}(V)}$$

Classically

$$\mathbb{V}(C,P)(\mathfrak{g},\ell,W)) = \frac{V_\ell(W)}{\mathfrak{g}_{C\setminus P}}$$

$$\begin{array}{c} \mathcal{V} \\ \text{sheaf} \downarrow \\ C \end{array} \qquad \nabla : \mathcal{V} \to \mathcal{V}\otimes\omega$$

$$\begin{array}{c} H^0\left(C\setminus P, \dfrac{\mathcal{V}\otimes\omega}{\nabla\mathcal{V}}\right) = \mathcal{L}_{C\setminus P}(V) \\ \downarrow \\ H^0\left(D_P^\times, \dfrac{\mathcal{V}\otimes\omega}{\nabla\mathcal{V}}\right) \\ \cong \\ \dfrac{V\otimes\mathbb{C}((t))}{\nabla} \circlearrowright M \end{array}$$



$$At^n = A^M_{(n)}$$

Theorem [D-Gibney-Tarasca]

When (C, P) varies in $\overline{\mathcal{M}}_{g,1}$ the spaces $\mathbb{V}((C, P_\bullet)(V; M))$ fit together to define a QUASI COHERENT SHEAF on $\overline{\mathcal{M}}_{g,1}$ denoted $\mathbb{V}_g(V; M)$ and called SHEAF of COINVARIANTS.

known for $g = 0$
& partial results for $g = 1$

Properties

- $\overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,n}$
- Proj. connection on $\mathcal{M}_{g,n}$

$\Rightarrow$
- vector bundle?
- rank \ chern classes?

- Glob. gen ?

Theorem [D-Gibney-Tarasca]

Let V be a VOA of CohFT-type
and $M_1, \ldots, M_n$ be simple V-modules
then $\mathbb{V}_g(V; M_\bullet)$ is a vector bundle of finite rank over $\overline{\mathcal{M}}_{g,n}$
and its Chern character defines a semisimple CohFT.

Rec. Thm: determined by $\mathcal{M}_{g,n}$
on $\mathcal{M}_{g,n}$ we have proj. connection
$\longrightarrow$ C_1 & rank

VOA of CohFt-type

(S) self dual: $V = \bigoplus V_i \cong \bigoplus V_i^\vee = V'$

(F) finiteness: C_2 or L_1 finiteness

(R) Rational: cat of V-repr. is semisimple
& finitely many simple modules

Idea of the proof

① $\mathbb{V}$ coherent : only need (F)

$\mathbb{V}$ quotient of coherent sheaf

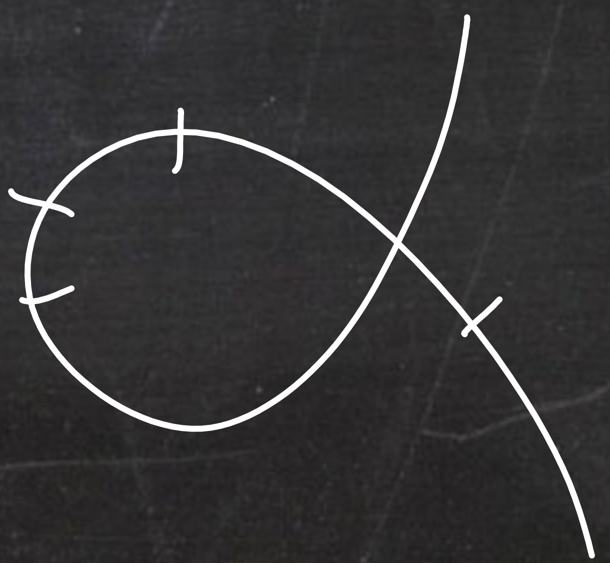
[Nagatomo Tsuchiya]

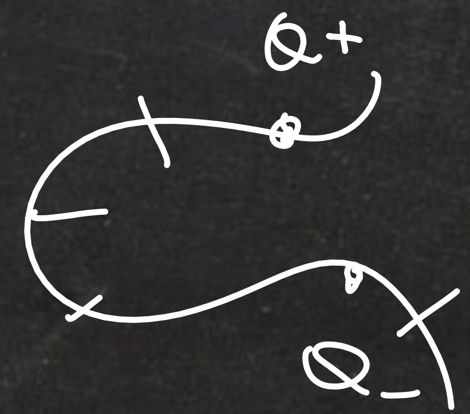
+

proj conn on $\mathcal{M}_{g,n}$ $\Rightarrow$ $\mathbb{V}_g$ loc. free over $\mathcal{M}_{g,n}$

② Boundary? FACTORIZATION RULE (sewing theorem)

$$\mathbb{V}((C, P_\cdot)(V; M_\cdot)) \cong \bigoplus_{W \text{ simple}} \mathbb{V}\left(C_N \begin{array}{c} P_\cdot \\ Q_+ \\ Q_- \end{array}\right)\left(V, \begin{array}{c} M_\cdot \\ W \\ W' \end{array}\right)$$





can compute rank from $\mathbb{V}_0(\mathbb{P}^1\, 0\, 1\, \infty, A B C)$

Properties that determine a CohFT

$$\Omega_{g,n} : M_1 \dots M_n \mapsto Ch(\mathbb{V}_g(V, M_1 \dots M_n)) \in H^*(\overline{M}_{g,n}, \mathbb{Q})$$

- Prop. of vacua $\quad \mathbb{V}(M_1 \dots M_n) = \mathbb{V}(M_1 \dots M_n V \dots V)$

(F,R) ∘ FACTORIZATION RULES

(S) ∘ $\mathbb{V}(\mathbb{P}^1, 0, 1, \infty, V, M, W) = \begin{cases} \mathbb{C} & \text{if } M = W' \\ 0 & \text{otherwise} \end{cases}$

Conclusion

To compute the Chern classes of coinvariants it's enough to know:

- $\operatorname{rank} \mathbb{V}(\mathbb{P}^1\ 0\ 1\ \infty, A\ B\ C)$ for all A, B, C
- c_V central charge of V
- a_M conf. weight for all simple M

First Chern Class

c_V = central charge of V

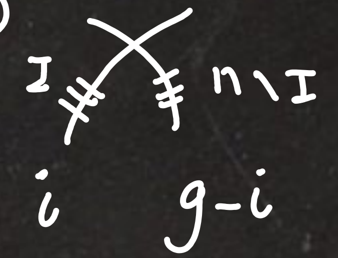
a_M = conformal dimension of M

$\lambda = c_1(\det R\pi_* \omega_C)$

$\psi_i = c_1(\sigma_i^* \omega_C)$

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "\sigma_i"'] \end{tikzcd}$$

$$c_1(\mathbb{V}_g(V, M_\bullet)) = \operatorname{rank}\mathbb{V}_g(V, M_\bullet)\left(\frac{c_V}{2}\lambda + \sum_{i=1}^{n} a_{M_i}\psi_i\right) - b_{irr}\,\delta \quad - \sum_{i,I} b_{i,I}\,\delta$$



- $b_{irr} = \sum_{W \in \mathcal{W}} a_W \operatorname{rank}\mathbb{V}_{g-1}(V, M_\bullet, W, W')$

- $b_{iI} = \sum_{W \in \mathcal{W}} a_W \operatorname{rank}\mathbb{V}_i(V; M_I, W) \cdot \operatorname{rank}\mathbb{V}_{g-i}(V; M_{n\setminus I}, W')$

Global Generation... where the analogy breaks!

There exist Vector bundles of coinv. on $\overline{M}_{0,4} = \mathbb{P}^1$ which are NOT globally gen, since deg < 0.

* Virasoro v.b. rank 2 deg -1
* Lattices line bundles deg $= -k$ for all $k \geq 0$

Theorem [D-Gibney]

Let V be a V.O.A generated in degree one and $M_1 \dots M_n$ simple V-modules, then $\mathbb{V}_0(V; M_\bullet)$ is globally generated over $\overline{M}_{0,n}$

Thank you!

Danke!