Moduli spaces and groups representations

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Intro on group actions

Let $(\mathbb{Z}, +)$ be the additive group, seen naturally as a subgroup of $(\mathbb{R}, +)$. The morphism

$$\mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \qquad (r, n) \mapsto r + n$$

gives an action of $\mathbb{Z}$ on $\mathbb{R}$.
We can then consider the quotient $\mathbb{R}/\mathbb{Z}$, i.e. the quotient of $\mathbb{R}$ by the equivalence relation

$$a \sim b \iff a = b + n \quad \text{for some } n \in \mathbb{Z}.$$

If we interpret this phenomenon topologically, we can identify $\mathbb{R}/\mathbb{Z}$ with the circle S^1.
The canonical projection map $\mathbb{R} \to S^1$ is the universal covering, realizing $\mathbb{Z}$ as fundamental group of S^1.

Twisted groups

Consider $\mathrm{SL}_r(\mathbb{C})$ and the cyclic group $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$. The map

$$\mathrm{SL}_r(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}_r(\mathbb{C}), \qquad (M, -1) \mapsto ((\overline{M})^\dagger)^{-1}$$

gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{SL}_r(\mathbb{C})$. Instead of taking the quotient, we can consider the invariant elements:

$$\mathrm{SL}_r(\mathbb{C})^{\mathbb{Z}/2\mathbb{Z}} = \{M \mid M^{-1} = \overline{M}^\dagger\} = \mathrm{SU}_r.$$

Generalizations

- Replacing $\mathbb{C}$ with any other domain R with an involution $a \mapsto \overline{a}$. For example $R = \mathbb{C}[t]$ with $\overline{t} = -t$.
- Considering $\widetilde{X} \to X$ Galois covering of curves and similarly defining the $\mathbb{Z}/2\mathbb{Z}$ invariants of $\mathrm{SL}_r(\widetilde{X})$.

Toy example

Consider $\mathrm{SL}_2(\mathbb{C})$ and its subgroup B of uppertriangular matrices. Given the actions

$$\mu\colon \mathrm{SL}_2(\mathbb{C}) \times B \times \mathrm{SL}_2(\mathbb{C}), \qquad (M, N) \mapsto MN$$

and (for $n \in \mathbb{Z}$)

$$\chi_n\colon B \times \mathbb{C} \to \mathbb{C}, \qquad (\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}, z) \mapsto a^n z$$

we can construct the quotients

$$P := \mathrm{SL}_2(\mathbb{C})/B \qquad \text{and} \qquad \mathcal{O}(n) := \left(\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}\right)/B$$

where the latter is obtained using

$$(M, z) \sim (MN, \chi_n^{-1}(N)z) \qquad N \in B.$$

Toy example

There is a canonical map

$$\pi_n \colon \mathcal{O}(n) \to P$$

whose fibres are isomorphic to $\mathbb{C}$. This construction defined a *line bundle* on P.

Theorem (Grothendieck)

All line bundles of P are described in this way.

Theorem (B-W,B)

For all $n \geq 0$, the set of sections of π_n is the standard representation of $SL_2(\mathbb{C})$ of dimension $n + 1$.

Conclusion: we can use representation theory to create new geometric objects and to study them.

Moduli Spaces

We see that P can also be defined as a space solving a moduli problem.

Moduli space: space which parametrizes algebro/geometric objects of the same type (up to isomorphism).

Example

Parametrizing spheres in $\mathbb{R}^3$.

$$\begin{array}{ll} \text{center} & (x,y,z) \in \mathbb{R}^3 \\ \text{radius} & r \in \mathbb{R}_{>0} \end{array} \implies \mathbb{R}^3 \times \mathbb{R}_{>0} \text{ is the moduli space}$$

P as moduli space

Claim: P parametrizes 1-dimensional vector spaces of $\mathbb{C}^2$. Observe that

$$0 \neq v \in \mathbb{C}^2 \qquad \text{gives a subspace } \langle v \rangle \subset \mathbb{C}^2$$

and for all $\lambda \neq 0$ we have that $\langle \lambda v \rangle = \langle v \rangle$. Hence

$$\mathbb{P}^1 := (\mathbb{C}^2 \setminus 0) \sim \qquad \text{where } v \sim w \iff v = \lambda w$$

is the space solving this moduli problem.

It is easy to check that the map

$$\mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 \setminus 0, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [a, c]$$

induces an isomorphism between P and $\mathbb{P}^1$:

Bun_{SL_r}

Fix now a smooth curve X of genus g and consider the space $\mathrm{Bun}_{\mathsf{SL}_r}$ parametrizing vector bundles on X of rank r and having trivial determinant. This means that we associate to each point of X a complex vector bundle of dimension r and that they are glued together via an element of SL_r.

We can describe the points of this moduli space as a double quotient. Fix $P \in X$, then

$$\mathrm{Bun}_G(\mathbb{C}) = \mathsf{SL}_r(X \setminus P) \setminus \mathsf{SL}_r(\mathbb{C}((t)))/\mathsf{SL}_r(\mathbb{C}[[t]])$$

which has this intuitive meaning: every SL_r-bundle is trivialized on $X \setminus P$ and on a small disk around P, so only $\mathsf{SL}_r(\mathbb{C}(t))$ tells us how to glue them on the intersection.

Line bundles of Bun_{SL_r}

This description is fundamental because it was the key observation to show that

Theorem

Line bundles of Bun_{SL_r} *are in bijection with* $\mathbb{Z}$*;*

Theorem (Beauville-Laszlo, Faltings)

Let $\ell \in \mathbb{N}$*. The space of sections of* $\mathcal{O}(\ell)$ *is canonically isomorphic to a vector space which naturally asises from representations attached to a central extension of* $SL_r(\mathbb{C}((t)))$*.*

We now want to understand what this space is.

Conformal blocks

Conformal blocks (attached to SL_r and of level ℓ) are finite dimensional complex vector spaces

$$\mathbb{V}_\ell((X, \underline{P}), (\mathrm{SL}_r, \underline{V}))$$

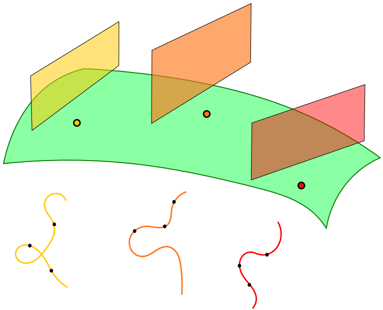
associated with two types of data:

- Geometry: A stable pointed curve $(X, P_1, \ldots P_n)$.
- Representation theory: n irreducible representations $V_1, \ldots V_n$ of SL_r of level at most ℓ.

To understand the importance of these objects, we need to introduce the next moduli space: $\overline{\mathcal{M}}_{g,n}$ parametrizes stable pointed curves of genus g.

Sheaves of conformal blocks on $\overline{\mathcal{M}}_{g,n}$

Fix n representations $\underline{V}$ of SL_r.



Theorem (TUY)

Associating to each pointed curve $(X, \underline{P})$ the conformal block $\mathbb{V}_\ell((X,\underline{P}),(SL_r,\underline{V}))$ defines a vector bundle

$$\mathbb{V}_\ell(\underline{V}) \text{ on } \overline{\mathcal{M}}_{g,n}.$$

$\Rightarrow$ As long as X and X' have the same genus

$$\dim \mathbb{V}_\ell((X,\underline{P}),(\mathrm{SL}_r,\underline{V})) = \dim \mathbb{V}_\ell((X',\underline{P}),(\mathrm{SL}_r,\underline{V}))$$

Relation with $\mathrm{Bun}_{\mathsf{SL}_r}$

Theorem (TUY)

When inserting the trivial representation, the bundle $\mathbb{V}_\ell(\underline{\mathbb{C}})$ is independent of the points chosen on the curves, i.e. it descends to a bundle on $\overline{\mathcal{M}}_g$. The fibers over a curve X are simply denoted $\mathbb{V}_\ell(X)$.

We can rephrase the theorem on sections of $\mathcal{O}(\ell)$ as:

Theorem (B-L,F)

The space of sections of $\mathcal{O}(\ell)$ is isomorphic to $\mathbb{V}_\ell(X)$.

$\Rightarrow$ We can compute the dimension of the space of global sections of $\mathcal{O}(\ell)$ independently of the curve we started with.

We need to understand if there is a better curve where to carry the computation!

Factorization

Let (X, P) be a curve with only one node Q. Then the normalization X^N is canonically marked by three poins: P, Q_+ and Q_-. Under this assumptions

Theorem (TUY)

$\mathbb{V}_\ell((X,P),(SL_r,V)) = \bigoplus_W \mathbb{V}_\ell((X^N,P,Q_+,Q_-),(SL_r,V,W,W^*))$

$\Rightarrow$ If we start with a nodal curve, we can reduce the computation to curves of lower genus: it is then enough to compute it on the case of $X = \mathbb{P}^1$ with three marked points.

Using this method it was possible to exhibit an explicit formula for $\mathbb{V}_\ell(X)$: the Verlinde formula [Faltings].

Generalizations

I generalized the construction of conformal blocks to the case of twisted groups $\mathcal{H} = \mathrm{SL}_r(\widetilde{X})^{\mathbb{Z}/2\mathbb{Z}}$ arising from Galois coverings of curves $\widetilde{X} \to X$.

Twisted conformal blocks are finite dimensional complex vector spaces

$$\mathbb{V}_\ell((\widetilde{X} \to X, \underline{P}), (\mathcal{H}, \underline{\mathcal{V}}))$$

associated with two types of data:

- Geometry: A stable covering of curves $(\widetilde{X} \to X, P_1, \dots P_n)$.
- Representation theory: n irreducible representations $\mathcal{V}_i$ of $\mathcal{H}_\ell(V)|_{P_i}$ of level at most ℓ.

Properties of Twisted Conformal Blocks [D.]

They satisfy similar properties to the classical ones:

- They fit together to define a vector bundle $\mathbb{V}_\ell(\underline{\mathcal{V}})$ on the stack $\overline{\mathcal{H}ur}_{g,n}$ parametrizing coverings of curves;
- When they depend on the trivial representation only, they descend to bundles on $\overline{\mathcal{H}ur}_g$.
- Factorization rules still hold:

$$\mathbb{V}_\ell((\widetilde{X} \to X, P), \mathcal{V}) = \bigoplus_{\mathcal{W}} \mathbb{V}_\ell((\widetilde{X}^N \to \widetilde{X}, P, Q_+, Q_-), (\mathcal{V}, \mathcal{W}, \mathcal{W}^*))$$

A couple of open questions

- Similarly to the case of SL_r bundles, also in this case line bundles $\mathrm{Bun}_{\mathcal{H}}$ has been studied, but it is more complicate.

  We expect that global sections of line bundles will be described by appropriate twisted conformal blocks associated to trivial representation.

- Computing a twisted Verlinde formula for these bundles will need a better understanding of degeneration of coverings and representations of $\mathcal{H}$.