Commun. Math. Phys. (2025) 406:131
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-025-05237-1

Communications in
Mathematical Physics

Conformal Blocks on Smoothings via Mode Transition Algebras

Chiara Damiolini[1], **Angela Gibney**[2], **Daniel Krashen**[2]

[1] Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA.
E-mail: chiara.damiolini@austin.utexas.edu
[2] Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA.
E-mail: agibney@math.upenn.edu; dkrashen@math.upenn.edu

Received: 21 April 2024 / Accepted: 30 December 2024

Abstract: Here we introduce a series of associative algebras attached to a vertex operator algebra V of CFT type, called mode transition algebras, and show they reflect both algebraic properties of V and geometric constructions on moduli of curves. Pointed and coordinatized curves, labeled by admissible V-modules, give rise to sheaves of coinvariants. We show that if the mode transition algebras admit multiplicative identities satisfying certain natural properties (called strong identity elements), these sheaves deform as wanted on families of curves with nodes. This provides new contexts in which coherent sheaves of coinvariants form vector bundles. We also show that mode transition algebras carry information about higher level Zhu algebras and generalized Verma modules. To illustrate, we explicitly describe the higher level Zhu algebras of the Heisenberg vertex operator algebra, proving a conjecture of Addabbo–Barron.

Mode transition algebras, introduced here, are a series of associative algebras that give insight into algebraic structures on moduli of stable pointed curves, and representations of the vertex operator algebras from which they are derived.

Admissible modules over a vertex operator algebras (VOAs for short) give rise to vector bundles of coinvariants on moduli of smooth, pointed, coordinatized curves [FBZ04]. To extend these to singular curves, coinvariants must deform as expected on smoothings of nodes, maintaining the same rank for singular curves as for smooth ones. By Theorem 5.0.3, this holds when coinvariants form coherent sheaves and the mode transition algebras (defined below) admit multiplicative identities with certain properties. Consequently, by Corollary 5.2.6 one obtains a potentially rich source of vector bundles, including as in Remark 3.4.6 (b), the well-known class given by rational and C_2-cofinite VOAs [TUY89,BFM91,NT05,DGT24], and by Corollary 7.4.1, a new family on moduli of stable pointed rational curves from modules over the Heisenberg VOA, which is neither C_2-cofinite nor rational. Vector bundles are valuable—their characteristic classes, degeneracy loci, and section rings have been instrumental in the understanding of moduli of curves (e.g. [HM82,Mum83,EH87,ELSV01,Far09,BCHM10]).

Published online: 07 May 2025

Known as essential to the study of the representation theory of VOAs, basic questions about the structure of higher level Zhu algebras remain open. Via Theorem 6.0.1, the mode transition algebras also give a new perspective on these higher level Zhu algebras. As an application, we prove [AB23a, Conjecture 8.1], thereby giving an explicit description of the higher level Zhu algebras for the Heisenberg VOA. This is done in Sect. 7 by analyzing the mode transition algebras associated to this VOA.

To describe our results more precisely, we set a small amount of notation, with more details given below. We first of all assume that V is a vertex operator algebra of CFT type. While they have applications in both VOA theory and algebraic geometry, we begin by describing the geometric problem, which motivated the definition of the mode transition algebras. By [DGK22], the sheaves of coinvariants are coherent when defined by modules over a C_2-cofinite VOA. By [DG23], coherence is also known to hold for some sheaves given by admissible modules of VOAs that are C_1-cofinite and not C_2-cofinite. It is natural then to ask when such coherent sheaves are vector bundles, as they were shown to be if V is both C_2-cofinite and rational [TUY89,BFM91,NT05,DGT24].

One may check that coherent sheaves are locally free by proving they are flat by virtue of Grothendieck's valuative criteria of flatness. The standard procedure first carried out by [TUY89, Theorem 6.2.1], and then followed in [NT05, Theorem 8.4.5], and [DGT24, VB Corollary]), is to argue inductively, using the factorization property ([TUY89, Theorem 6.2.6], [NT05, Theorem 8.4.3], [DGT24, Theorem 7.0.1], [DGK22, Theorem 3.2]). However, here factorization is not available, since we do not assume that Zhu's algebra $A(V)$ is finite and semi-simple ([DGK22, Proposition 7.1]).

Instead, as explained in the proof of Corollary 5.2.6, the geometric insight here, is that in place of factorization, one may show that the dimension of spaces of coinvariants remain constant as nodes are smoothed in families. This relies on the mode transition algebras admitting multiplicative identities that also act as identity elements on appropriate modules (we call these *strong identity elements* in Definition/Lemma 3.3.1). The base case then follows from the assumption of coherence and that sheaves of coinvariants support a projectively flat connection on moduli of smooth curves [FBZ04,DGT21]. We refer to this process as smoothing of coinvariants. Let us now summarize the strategy.

For simplicity, let $\mathscr{C}_0$ be a projective curve over $\mathbb{C}$ with a single node Q, n smooth points $P_\bullet = (P_1, \dots, P_n)$, and formal coordinates $t_\bullet = (t_1, \dots, t_n)$ at $P_\bullet$. Let $W^1, \dots, W^n$ be an n-tuple of admissible V-modules, or equivalently of smooth $\mathscr{U}$-modules, where $\mathscr{U}$ is the universal enveloping algebra associated to V (defined in detail in Sect. 2).

The vector space of coinvariants $[W^\bullet]_{(\mathscr{C}_0, P_\bullet, t_\bullet)}$ is the largest quotient of $W^\bullet = W^1 \otimes \cdots \otimes W^n$ on which the Chiral Lie algebra $\mathcal{L}_{\mathscr{C}_0 \setminus P_\bullet}(V)$ acts trivially (described here in Sect. 4). Coinvariants at $\mathscr{C}_0$ are related to those on the normalization $\eta : \widetilde{\mathscr{C}}_0 \to \mathscr{C}_0$ of $\mathscr{C}_0$ at Q. Namely, by [DGT24] the map

$$\alpha_0 : W_0^\bullet \to W_0^\bullet \otimes \mathsf{A}, \qquad u \mapsto u \otimes 1^{\mathsf{A}},$$

gives rise to an $\mathcal{L}_{\mathscr{C}_0 \setminus P_\bullet}(V)$-module map, where $\mathsf{A}_0(V) := \mathsf{A}$ denotes the (level zero) Zhu algebra of V. This induces a map between spaces of coinvariants

$$[\alpha_0] : [W^\bullet]_{(\mathscr{C}_0, P_\bullet, t_\bullet)} \to [W^\bullet \otimes \Phi(\mathsf{A})]_{\left(\widetilde{\mathscr{C}}_0, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm\right)}.$$

Here $\Phi(\mathsf{A})$ is a $\mathscr{U}$-bimodule assigned at points $Q_\pm$ lying over $\eta^{-1}(Q)$, and $s_\pm$ are formal coordinates at $Q_\pm$. By [DGK22], the map $[\alpha_0]$ is an isomorphism if V is C_1-cofinite.

One may extend the nodal curve $\mathscr{C}_0$ to a smoothing family $(\mathscr{C}, P_\bullet, t_\bullet)$ over the scheme $S = \mathrm{Spec}(\mathbb{C}[\![q]\!])$, with special fiber $(\mathscr{C}_0, P_\bullet, t_\bullet)$, and smooth generic fiber, while one may trivially extend $\widetilde{\mathscr{C}}_0$ to a family $(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)$ over S. While the central fibers of these two families of curves are related by normalization, there is no map between $\widetilde{\mathscr{C}}$ and $\mathscr{C}$. However, for V rational and C_2-cofinite, sheaves of coinvariants on S are naturally isomorphic, an essential ingredient in the proof that such sheaves are locally free under these assumptions [DGT24].

To obtain an analogous isomorphism of coinvariants under less restrictive conditions, our main idea is to generalize the algebra structure of $X_d \otimes X_d^\vee \subset \Phi(\mathsf{A})$, which exists for simple admissible V-modules $X = \bigoplus_d X_d$, and for all $d \in \mathbb{N}$ (see Remark 3.4.6). Namely, we show that $\Phi(\mathsf{A})$ has the structure of a bi-graded algebra, which we call the *mode transition algebra* and denote $\mathfrak{A} = \bigoplus_{d_1,d_2 \in \mathbb{Z}} \mathfrak{A}_{d_1,d_2}$. We show that $\mathfrak{A}$ acts on generalized Verma modules $\Phi^{\mathsf{L}}(W_0) = \bigoplus_d W_d$, such that the subalgebras $\mathfrak{A}_d := \mathfrak{A}_{d,-d}$, which we refer to as the *d-th mode transition algebras*, act on the degree d components W_d (these terms are defined in Sect. 3.1 and Sect. 3.2).

We say that V satisfies smoothing (Definition 5.0.1), if for every pair $(W^\bullet, \mathscr{C}_0)$, consisting of n admissible V-modules $W^\bullet$ and a stable n-pointed curve $\mathscr{C}_0$ with a node, there exist an element $\mathscr{I} = \sum_{d \geq 0} \mathscr{I}_d q^d \in \mathfrak{A}[\![q]\!]$, such that the map

$$\alpha \colon W^\bullet[\![q]\!] \longrightarrow (W^\bullet \otimes \mathfrak{A})[\![q]\!], \qquad u \mapsto u \otimes \mathscr{I},$$

is an $\mathcal{L}_{\mathscr{C} \setminus P_\bullet}(V)$-module homomorphism which extends α_0.

In Theorem 5.0.3, we equate smoothing for V with a property of multiplicative identity elements in the d-th mode transition algebras $\mathfrak{A}_d$, when they exist. Specifically, if $\mathfrak{A}_d$ admit identity elements $\mathscr{I}_d \in \mathfrak{A}_d$ for all $d \in \mathbb{N}$, satisfying any of the equivalent properties of Definition/Lemma 3.3.1, then we say that $\mathscr{I}_d \in \mathfrak{A}_d$ is a *strong identity element*.

Theorem (5.0.3). *Let V be a VOA of CFT-type. The algebras $\mathfrak{A}_d = \mathfrak{A}_d(V)$ admit strong identity elements for all $d \in \mathbb{N}$ if and only if V satisfies smoothing.*

We remark that the analogue to α is called the sewing map in [TUY89,NT05,DGT24]. As an application of Theorem 5.0.3 we obtain geometric consequences stated as Corollary 5.2.1 and Corollary 5.2.6. A particular case of which is as follows:

Corollary (5.2.6). *If V is C_2-cofinite and satisfies smoothing, then $\mathbb{V}(V; W^\bullet)$ is a vector bundle on $\overline{\mathcal{M}}_{g,n}$ for simple V-modules $W^1, \ldots, W^n$.*

By Remark 3.4.6, rational VOAs satisfy smoothing, so Corollary 5.2.6 specializes to [DGT24, VB Corollary]. As is shown in Corollary 7.4.1, one can apply the full statement of Corollary 5.2.6 to show that modules over VOAs derived from Heisenberg Lie algebras of dimension one (which are C_1-cofinite, but neither C_2-cofinite nor rational), define vector bundles on moduli of stable pointed rational curves (see Corollary 7.4.1).

Theorem 6.0.1, described next, gives further tools for investigating other VOAs which may or may not satisfy smoothing by providing information about the relationship between mode transition algebras and higher level Zhu algebras and their representations.

Recall that in [Zhu96] Zhu defines an induction functor, which takes $\mathsf{A} = \mathsf{A}(V)$-modules to V-modules. In [DLM98], this construction is described as a two-step process, where from an A module one first forms a "generalized Verma module." In Definition 3.1.1 we define a different construction of this generalized Verma module functor, which we denote by Φ^{L}. This is a crucial ingredient in this work. Through his functor, Zhu shows that there is a bijection between simple A-modules and simple V-modules. As we explain in Remark 3.4.6 if V is rational, then our functor Φ^{L} agrees with Zhu's induction functor, therefore giving a correspondence between simple A-modules and simple V-modules. However, if A is either not finite-dimensional or is not semi-simple, then there may be indecomposable, but non-simple V-modules that are not induced from simple indecomposable modules over A via Φ^{L}. To describe such modules not induced from A, [DLM98] defined the higher level Zhu algebras A_d for $d \in \mathbb{N}$, further studied in [BVWY19a].

The mode transition algebras $\mathfrak{A}_d$ are related to the higher level Zhu algebras A_d. For instance, $\mathfrak{A}_0 = \mathsf{A}_0 = \mathsf{A}$ (Remark 3.2.4), and by Lemma B.3.1, there is an exact sequence

$$\mathfrak{A}_d \xrightarrow{\mu_d} \mathsf{A}_d \xrightarrow{\pi_d} \mathsf{A}_{d-1} \longrightarrow 0. \tag{1}$$

When $\mathfrak{A}_d$ admits an identity element (and not necessarily a strong identity element), μ_d is injective and the sequence splits by Part (a) of Theorem 6.0.1. In particular, if V is C_2-cofinite, as observed in [Buh02,GN03,Miy04,He17], the d-th Zhu algebras A_d are finite dimensional, hence if $\mathfrak{A}_d$ has an identity element, it is finite dimensional as well.

We note that (1) may be exact when $\mathfrak{A}_d$ does not admit an identity element. For instance, in Sect. 8 we show exactness of (1) when $d = 1$ for the Virasoro VOA Vir_c, for any values of c, and use it to show that $\mathfrak{A}_1$ does not admit an identity element. In particular, by Theorem 5.0.3, one finds that Vir_c never satisfies smoothing. When c is in the discrete series, the maximal simple quotient L_c of Vir_c is rational and C_2-cofinite, hence by Remark 3.4.6 (b) (or by [DGT24]) it will satisfy smoothing.

Theorem 6.0.1 allows one to use the mode transition algebras to obtain other valuable structural information about the higher level Zhu algebras.

Theorem (6.0.1). *(a) If the d-th mode transition algebra $\mathfrak{A}_d$ admits an identity element, then (1) is split exact, and $\mathsf{A}_d \cong \mathfrak{A}_d \times \mathsf{A}_{d-1}$ as rings. In particular, if $\mathfrak{A}_d$ admits an identity element for every $d \in \mathbb{Z}_{\geq 0}$ then $\mathsf{A}_d \cong \mathfrak{A}_d \oplus \mathfrak{A}_{d-1} \oplus \cdots \oplus \mathfrak{A}_0$.*

(b) For $\mathfrak{A} = \mathfrak{A}(V)$, if $\mathfrak{A}_d$ admits a strong identity for all $d \in \mathbb{N}$, so that smoothing holds for V, then given any generalized Verma module $W = \Phi^{\mathsf{L}}(W_0) = \bigoplus_{d \in \mathbb{N}} W_d$ where L_0 acts on W_0 as a scalar with eigenvalue $c_W \in \mathbb{C}$, there is no proper submodule $Z \subset W$ with $c_Z - c_W \in \mathbb{Z}_{>0}$ for every eigenvalue c_Z of L_0 on Z.

We refer to Sect. 3.1 for a discussion about generalized Verma modules.

We note that by Sect. B.3.1 and Lemma B.3.3, the exact sequence (1) as well as Part *(a)* of Theorem 6.0.1 hold for generalized higher Zhu algebras and generalized d-th mode transition algebras (see Definition B.1.1 and Definition B.2.6). For further discussion see Sect. 9.5.

We now describe some further consequences of Theorem 5.0.3 and Theorem 6.0.1.

In Sect. 7 we describe the d-th mode transition algebras $\mathfrak{A}_d$ for the Heisenberg VOA $M_a(1)$ defined by a one dimensional Heisenberg Lie algebra (denoted π in [FBZ04]), and show the $\mathfrak{A}_d$ admit strong identity elements for all $d \in \mathbb{N}$. In particular, Theorem 6.0.1 and Proposition 7.2.1 imply that the conjecture of Addabbo and Barron [AB23a, Conj 8.1] holds, and one can write

$$\mathsf{A}_d(\pi) = \mathsf{A}_d(M_a(1)) \cong \prod_{j=0}^{d} \mathrm{Mat}_{p(j)}(\mathbb{C}[x]), \tag{2}$$

where $p(j)$ is the number of ways to decompose j into a sum of positive integers, with $p(0) = 1$. The level one Zhu algebra $A_1(M_a(1))$ was first constructed in the paper [BVWY19b], and then later announced in [BVWY19a]. In [AB23b] the authors determine $A_2(M_a(1))$ using the infrastructure for finding generators and relations for higher level Zhu algebras they had developed in [AB23a].

To illustrate their computability, we note that in [DGK24] the d-th mode transition algebras for VOAs derived from Heisenberg Lie algebras of arbitrary rank are shown to admit strong identities for all d, and further results developed there allow us to compute higher Zhu algebras for these and other examples.

In Sect. 9.1.2, we use Part (b) of Theorem 6.0.1 to show that the family of triplet vertex operator algebras $\mathcal{W}(p)$ does not satisfy smoothing for every $p \geq 2$. We do this by giving an explicit pair of modules $W \subset Z$ with $W = \Phi^{\mathsf{L}}(W_0)$ and such that $c_Z > c_W$. The actual pair of modules used was suggested to us by Thomas Creutzig (with some details filled in by Simon Wood). Dražen Adamović had also sketched for us the existence of such an example. The importance of this example is that it establishes that smoothing is not guaranteed to hold for a C_2-cofinite VOA if rationality is not assumed. In particular, while sheaves of coinvariants defined by the representations of C_2-cofinite VOAs are coherent, this can be seen as an indication that they may not necessarily be locally free. Taken together with the family of Heisenberg vector bundles from Corollary 7.4.1, this example illustrates the subtlety of the problem of determining which sheaves of coinvariants define vector bundles.

Since $\mathrm{SF}_1^+ = \mathcal{W}(2)$, we also expect that smoothing will not hold for the family of symplectic fermion algebras SF_d^+, for all d, which are C_2-cofinite and not rational. It is natural to ask whether there is an example of a vertex operator algebra that is C_2-cofinite, is not rational, and satisfies smoothing (see Sect. 9.1).

Finally, we emphasize that our procedure to use smoothing to show that sheaves of coinvariants are locally free is just one approach to this problem (see Sect. 9.2).

Plan of the Paper

In Sect. 1, we set the terminology used here for vertex operator algebras and their representations. In Sect. 2, we provide detailed descriptions of the universal enveloping algebra $\mathscr{U}$ associated to a vertex operator algebra V. Technical details are given in Appendix A, where an axiomatic treatment of the constructions of the graded and filtered enveloping algebras as topological or semi-normed algebras is given. The concepts discussed involving filtered and graded completions can be found throughout the VOA literature (for instance in [TUY89, FZ92, FBZ04, Fre07, NT05, MNT10]), but little is said about how they relate to one another. We discuss these relations in Sect. 2. In Sect. 3 we give an alternative construction of the generalized Verma module functor Φ^{L} (and the right-analogue Φ^{R}) from the category of A-modules, to the category of smooth left (and right) $\mathscr{U}$-modules. We use a combination of Φ^{L} and Φ^{R} to define the mode transition algebras $\mathfrak{A}_d \subset \mathfrak{A}$. More general versions of these constructions are defined in Appendix B, where their analogous properties are proved. In Sect. 4, smoothing is formally defined, and we describe sheaves of coinvariants on families of pointed and coordinatized curves in general terms, and cite the relevant references. In Sect. 5 we prove Theorem 5.0.3, Corollary 5.2.1, and Corollary 5.2.6. In Sect. 6 we prove Theorem 6.0.1 Part *(b)*, while Part *(a)* is detailed in Appendix B. In Sect. 7 we compute the mode transition algebras $\mathfrak{A}_d$ for the Heisenberg algebra for all d. In Sect. 8 we compute the 1st mode transition algebras for the non-discrete series Virasoro VOAs. We ask a number of questions in Sect. 9. In Sect. 9.1 and in Sect. 9.2 questions are discussed about C_2-cofinite and non-rational VOAs that may not satisfy smoothing, and whether their induced sheaves of coinvariants may still define vector bundles. In Sect. 9.3 and Sect. 9.4, natural questions concerning the category of $\mathfrak{A}(V)$-modules and about the relation of $\mathfrak{A}(V)$ with associative algebras previously constructed in the literature are described. Finally, as noted, many of the results here are stated and proved for generalizations of higher level Zhu algebras and of mode transition algebras, and in Sect. 9.5 we raise the question of finding other examples and applications of such algebraic structures, beyond those naturally associated to a vertex operator algebra.

1. Background on VOAs and Their Modules

In Sect. 1.1 we state the conventions we follow for vertex operator algebras and their representations. Throughout this paper, by an algebra we mean an associative algebra which is not necessarily commutative and by a ring we mean an algebra over $\mathbb{Z}$. We refer to [FZ92, Zhu96, BFM91, NT05] for more details about vertex operator algebras and their modules.

1.1. VOAs and Their Representations. We recall here the definition of a vertex operator algebra of CFT type, which in the paper will be simply called a *VOA*.

Definition 1.1.1. A *vertex operator algebra of CFT-type* a is fourtuple $(V, \mathbf{1}, \omega, Y(\cdot, z))$:

(a) $V = \bigoplus_{i\in\mathbb{N}} V_i$ is a vector space with $\dim V_i < \infty$, and $\dim V_0 = 1$;
(b) $\mathbf{1}$ is an element in V_0, called the *vacuum vector*;
(c) ω is an element in V_2, called the *conformal vector*;
(d) $Y(\cdot, z)\colon V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ is a linear map $a \mapsto Y(a, z) := \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1}$. The series $Y(a, z)$ is called the *vertex operator* assigned to $a \in V$,

satisfying the following axioms:

(a) *(vertex operators are fields)* for all $a, b \in V$, $a_{(m)}b = 0$, for $m \gg 0$;
(b) *(vertex operators of the vacuum)* $Y(\mathbf{1}, z) = \mathrm{id}_V$, that is

$$\mathbf{1}_{(-1)} = \mathrm{id}_V \quad \text{and} \quad \mathbf{1}_{(m)} = 0, \quad \text{for } m \neq -1,$$

and for all $a \in V$, $Y(a, z)\mathbf{1} \in a + zV[\![z]\!]$, that is

$$a_{(-1)}\mathbf{1} = a \quad \text{and} \quad a_{(m)}\mathbf{1} = 0, \quad \text{for } m \geq 0;$$

(c) *(weak commutativity)* for all $a, b \in V$, there exists an $N \in \mathbb{N}$ such that

$$(z_1 - z_2)^N \, [Y(a, z_1), Y(b, z_2)] = 0 \quad \text{in } \mathrm{End}(V)[\![z_1^{\pm 1}, z_2^{\pm 1}]\!];$$

(d) *(conformal structure)* for $Y(\omega, z) = \sum_{m\in\mathbb{Z}} \omega_{(m)} z^{-m-1}$,

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p - q)\, \omega_{(p+q+1)} + \frac{c}{12}\, \delta_{p+q,0}\, (p^3 - p)\, \mathrm{id}_V.$$

Here $c \in \mathbb{C}$ is the *central charge* of V. Moreover:

$$\omega_{(1)}|_{V_m} = m \cdot \mathrm{id}_V, \quad \text{for all } m, \quad \text{and} \quad Y\left(\omega_{(0)}a, z\right) = \frac{d}{dz} Y(a, z).$$

Definition 1.1.2. An *admissible V-module* is a $\mathbb{C}$-vector space W together with a linear map

$$Y^W(\cdot, z)\colon V \to \mathrm{End}(W)[\![z, z^{-1}]\!], \; a \in V \mapsto Y^W(a, z) := \sum_{m\in\mathbb{Z}} a^W_{(m)} z^{-m-1},$$

which satisfies the following axioms:

(a) *(vertex operators are fields)* if $a \in V$ and $u \in W$, then $a^W_{(m)}u = 0$, for $m \gg 0$;
(b) *(vertex operators of the vacuum)* $Y^W(\mathbf{1}, z) = \mathrm{id}_W$;
(c) *(weak commutativity)* for all $a, b \in V$, there exists an $N \in \mathbb{N}$ such that for all $u \in W$

$$(z_1 - z_2)^N \left[Y^W(a, z_1), Y^W(b, z_2)\right] u = 0;$$

(d) *(weak associativity)* for all $a \in V$ and $u \in W$, there exists an $N \in \mathbb{N}$, such that for all $b \in V$, one has

$$(z_1 + z_2)^N \left(Y^W(Y(a, z_1)b, z_2) - Y^W(a, z_1 + z_2)Y^W(b, z_2)\right) u = 0;$$

(e) *(conformal structure)* for $Y^W(\omega, z) = \sum_{m\in\mathbb{Z}} \omega^W_{(m)} z^{-m-1}$, one has

$$\left[\omega^W_{(p+1)}, \omega^W_{(q+1)}\right] = (p - q)\, \omega^W_{(p+q+1)} + \frac{c}{12}\, \delta_{p+q,0}\, (p^3 - p)\, \mathrm{id}_W.$$

where $c \in \mathbb{C}$ is the central charge of V. Moreover $Y^W(L_{-1}a, z) = \frac{d}{dz} Y^W(a, z)$;
(f) ($\mathbb{N}$-gradability) W admits a grading $W = \bigoplus_{n\in\mathbb{N}} W_n$ with $a^W_{(m)} W_n \subset W_{n+\deg(a)-m-1}$.

As one can see in the literature, e.g. by [DL93, FHL93, LL04, Li96], weak associativity and weak commutativity together are equivalent to the *Jacobi identity*: for $\ell, m, n \in \mathbb{Z}$, and $a, b \in V$

$$\sum_{i\geq 0}(-1)^i \binom{\ell}{i} \left(a^W_{(m+\ell-i)} b^W_{(n+i)} - (-1)^\ell b^W_{(n+\ell-i)} a^W_{(m+i)}\right) = \sum_{i\geq 0} \binom{m}{i} (a_{(\ell+i)}(b))^W_{(m+n-i)}.$$

Moreover, by [DLM97, Lemma 2.2], axiom (1.1.2) is redundant.

Notation 1.1.3. *Throughout, whenever we say that V is a VOA we implicitly assume that V is of CFT-type. Similarly, when we say that W is a V-module, we implicitly assume that it is admissible, as in Definition 1.1.2.*

2. The Universal Enveloping Algebra of a VOA

Here we describe constructions of the universal enveloping algebra associated to a VOA V of CFT-type, as quotients of certain graded, as well as (left and right) filtered completions of the universal enveloping algebra of the Lie algebra associated to V. Filtered completions are essential to our constructions, as they are compatible with crucial restriction maps from the Chiral Lie algebra to certain ancillary Lie algebras, allowing for the definition of the action of the Chiral Lie algebra on (tensor products of) V-modules. The graded completion, on the other hand, allows both for ease in computation, and simpler descriptions of induced modules, and bimodules, and in Sect. 3.2 of the mode transition algebras. While these concepts are treated in one way or another throughout the VOA literature, for instance in [FZ92, FBZ04, Fre07, NT05, MNT10], we provide here and in Appendix A, a uniform description, where many details are given, clarifications are made, and the different constructions are compared to one another. We further remark that, although this section assumes that V is a vertex operator algebra, however all the arguments and construction here in Sect. 2 hold assuming only that V is an $\mathbb{N}$-graded vertex algebra since the conformal structure does not play a role (see also Sect. 9.5).

2.1. Graded and Filtered Completions. We recall the constructions of the universal enveloping algebra [FZ92] and the current algebra [NT05] associated to a VOA V.

2.2. Split Filtrations. The underlying vector spaces of the objects we will need to consider will either be graded or filtered (sometimes both), and these filtrations and gradings will be related to each other. The basic example of this is the space of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ which we choose to grade by $\mathbb{C}[t, t^{-1}]_n = \mathbb{C}t^{-n-1}$, and the space of Laurent series $\mathbb{C}((t))$, which admits an increasing filtration by setting $\mathbb{C}((t))_{\leq n} := t^{-n-1}\mathbb{C}[[t]]$. We will refer to this filtration as a *left filtration* of $\mathbb{C}((t))$ (see Definition A.1.1). In this situation, when we have a graded subspace of a filtered space $\mathbb{C}[t, t^{-1}] \subset \mathbb{C}((t))$ which identifies the degree n part of the graded subspace with the degree n part of the associated graded space, we say that this pair gives a *split filtration* (see Definition A.1.6). Similarly, we refer to the filtration of $\mathbb{C}((t^{-1}))$ given by $\mathbb{C}((t^{-1}))_{\geq n} = t^{-n-1}\mathbb{C}[[t^{-1}]]$ as a *right filtration*, and the pair $\mathbb{C}[t, t^{-1}] \subset \mathbb{C}((t^{-1}))$ is also a split filtration.

2.3. Lie Algebras. Since V is a VOA of CFT-type, it is $\mathbb{N}$ graded as a vector space, and so admits a trivial left (respectively right) split filtration, given by $V_{\leq n} = \bigoplus_{d\leq n} V_d$ (respectively $V_{\geq n} = \bigoplus_{d\geq n} V_d$). In view of Definition/Lemma A.3.3, tensor products of split-filtered modules are naturally split-filtered, and consequently

$$V \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \subset V \otimes_{\mathbb{C}} \mathbb{C}((t)) \quad \text{and} \quad V \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \subset V \otimes_{\mathbb{C}} \mathbb{C}((t^{-1}))$$

define splittings of their left and right filtrations.

Remark 2.3.1. Concretely, we can define the map $\mathsf{val}\colon V \otimes \mathbb{C}((t)) \to \mathbb{Z}$ by

$$\mathsf{val}(a \otimes f(t)) = \deg(a) - N - 1.$$

for a homogeneous element $a \in V$ and $f(t) \in t^N\mathbb{C}[[t]]\backslash t^{N-1}\mathbb{C}[[t]]$. The natural left filtration on $V \otimes_{\mathbb{C}} \mathbb{C}((t))$ is then given by $(V \otimes \mathbb{C}((t)))_{\leq n} := \mathsf{val}^{-1}(-\infty, n]$.

The linear map $\nabla = L_{-1} \otimes \mathrm{id} + \mathrm{id} \otimes \frac{d}{dt}$ is a linear endomorphism of each of these spaces of degree 1 (see Definition A.1.10). We define

$$\mathfrak{L}(V)^{\mathsf{L}} = \frac{V \otimes_{\mathbb{C}} \mathbb{C}((t))}{\mathrm{Im}(\nabla)}, \quad \mathfrak{L}(V)^{\mathsf{R}} = \frac{V \otimes_{\mathbb{C}} \mathbb{C}((t^{-1}))}{\mathrm{Im}(\nabla)}, \quad \text{and} \quad \mathfrak{L}(V)^{\mathsf{f}} = \frac{V \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]}{\mathrm{Im}(\nabla)}.$$

These have induced split filtrations via $\mathfrak{L}(V)^{\mathsf{f}} \subset \mathfrak{L}(V)^{\mathsf{L}}, \mathfrak{L}(V)^{\mathsf{R}}$ by Lemma A.1.14. These filtered and graded vector spaces admit (filtered and graded) Lie algebra structures, with Lie brackets defined by:

$$[a \otimes f(t), b \otimes g(t)] := \sum_{k\geq 0} \frac{1}{k!} \left(a_{(k)}(b)\right) \otimes g(t)\frac{d^k(f(t))}{dt^k},$$

for all $a, b \in V$ and $f(t), g(t) \in \mathbb{C}((t))$ or $f(t), g(t) \in \mathbb{C}((t^{-1}))$.

More concretely in the case of $\mathfrak{L}(V)^{\mathsf{f}}$, for $a \in V$ and $i \in \mathbb{Z}$ we denote by $a_{[i]}$ the class of the element $a \otimes t^i$ in $\mathfrak{L}(V)^{\mathsf{f}} \subset \mathfrak{L}(V)^{\mathsf{L}}, \mathfrak{L}(V)^{\mathsf{R}}$. The restriction of the Lie bracket on $\mathfrak{L}(V)^{\mathsf{f}}$ then is given by the following formula

$$\left[a_{[i]}, b_{[j]}\right] := \sum_{k\geq 0} \binom{i}{k} \left(a_{(k)}(b)\right)_{[i+j-k]},$$

for all $a, b \in V$ and $i, j \in \mathbb{Z}$. Extending the notation introduced in [DGT24], we call $\mathfrak{L}(V)^{\mathsf{L}}$ and $\mathfrak{L}(V)^{\mathsf{R}}$ the left and right *ancillary Lie algebras* and $\mathfrak{L}(V)^{\mathsf{f}}$ the *finite ancillary Lie algebra*. Note that $\mathfrak{L}(V)^{\mathsf{L}}$ is isomorphic to the *current Lie algebra* $\mathfrak{g}(V)$ from [NT05].

2.4. Enveloping Algebras. We now let $U^{\mathsf{L}}, U^{\mathsf{R}}, U$ be the universal enveloping algebras of $\mathfrak{L}(V)^{\mathsf{L}}, \mathfrak{L}(V)^{\mathsf{R}}, \mathfrak{L}(V)^{\mathsf{f}}$ respectively. These enveloping algebras are left filtered, right filtered, and graded respectively, and we note that $U \subset U^{\mathsf{L}}, U^{\mathsf{R}}$ again give splittings to the respective filtrations (see Lemma A.3.6). In the language of Definition A.9.1 we will say that $(U^{\mathsf{L}}, U, U^{\mathsf{R}})$ forms a good triple of associative algebras.

Example 2.4.1. These induced filtrations can be explicitly described. For instance we have that U_d is linearly spanned by elements $\ell^1 \cdots \ell^k$ such that $\ell^i \in \mathfrak{L}(V)^{\mathsf{f}}_{d_i}$ with $\sum_{i=1}^k d_i = d$ and $d_i \in \mathbb{Z}$ (with k possibly zero when $d = 0$). Analogously $(U^{\mathsf{L}})_{\leq d}$ is linearly spanned by elements $\ell^1 \cdots \ell^k$ such that $\ell^i \in \mathfrak{L}(V)_{\leq d_i}$ and $\sum_i d_i = d$ (and k possibly zero if $d \geq 0$).

These enveloping algebras have an additional structure of a topology induced by seminorms, which can be described in terms of systems of neighborhoods of 0 (as in Definition A.4.1 and Remark A.4.4). These neighborhoods of the identity are given by left ideals $\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}}$ of U^{L} and $\mathrm{N}^n_{\mathsf{L}} U$ of U and right ideals $\mathrm{N}^n_{\mathsf{R}} U^{\mathsf{R}}$ of U^{R} and $\mathrm{N}^n_{\mathsf{R}} U$ of U defined by:

$$\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}} = U^{\mathsf{L}} U^{\mathsf{L}}_{\leq -n}, \quad \mathrm{N}^n_{\mathsf{L}} U = U U_{\leq -n}, \quad \mathrm{N}^n_{\mathsf{R}} U^{\mathsf{R}} = U^{\mathsf{R}}_{\geq n} U^{\mathsf{R}}, \quad \mathrm{N}^n_{\mathsf{R}} U = U_{\geq n} U.$$

This definition coincides with that of a canonical seminorm on a (split-)filtered algebra, as described in Definition A.6.1, and in particular gives a good seminorm on the triple (see Definition A.9.3 and Remark A.9.7). Most useful for us is that the category of good triples of algebras with good seminorms is closed under quotients and completions (Corollary A.9.9 and Corollary A.9.10).

These neighborhoods can also be described alternately as follows:

Lemma 2.4.2. *One has an identification of left ideals* $\mathrm{N}^n_{\mathsf{L}} U = U U_{\leq -n} = U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$. *Similarly, one has* $\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}} = U^{\mathsf{L}} \mathfrak{L}(V)^{\mathsf{L}}_{\leq -n}$, $\mathrm{N}^n_{\mathsf{R}} U^{\mathsf{R}} = \mathfrak{L}(V)^{\mathsf{R}}_{\geq n} U^{\mathsf{R}}$ *and* $\mathrm{N}^n_{\mathsf{R}} U = \mathfrak{L}(V)^{\mathsf{f}}_{\geq n} U$.

We note that in Lemma 2.4.2, we identify $\mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$ with its image in U.

Proof. We will prove only the first equality, the others following by similar methods. Clearly $U U_{\leq -n}$ contains $U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$, so we need only prove the reverse inclusion, which amounts to showing $U_{-n} \subset U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$. For this, suppose we have an element $\alpha = a^1_{[n_1]} a^2_{[n_2]} \cdots a^r_{[n_r]} \in U_{-n}$. By inductively using [He17, Lemma A.2.1], for $M >> 0$, we may write $\alpha = a^1_{[n_1]} a^2_{[n_2]} \cdots a^r_{[n_r]} = \beta + \gamma$, with $\beta \in \mathfrak{L}(V)^{\mathsf{f}}_{-n}$ and $\gamma \in U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -M}$. In particular, choosing $M \geq n$, we find $\gamma \in U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$, showing $\alpha \in U \mathfrak{L}(V)^{\mathsf{f}}_{\leq -n}$ as desired. □

We can restrict these seminorms to various filtered and graded parts of these algebras as in Definition A.4.6. For example, we write $\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}}_{\leq p}$ to denote $\mathrm{N}^n_{\mathsf{L}}(U^{\mathsf{L}}) \cap U^{\mathsf{L}}_{\leq p}$. Concretely we obtain systems of neighborhoods as follows (see Lemma A.3.4 for more details):

$$\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}}_{\leq p} = (U^{\mathsf{L}} U^{\mathsf{L}}_{\leq -n})_{\leq p} = \sum_{j \leq -n} U^{\mathsf{L}}_{\leq p-j} U^{\mathsf{L}}_{\leq j}, \qquad \mathrm{N}^n_{\mathsf{R}} U^{\mathsf{R}}_{\geq p} = (U^{\mathsf{R}}_{\geq n} U^{\mathsf{R}})_{\geq p} = \sum_{i \geq n} U^{\mathsf{R}}_{\geq i} U^{\mathsf{R}}_{\geq p-i}$$

$$\mathrm{N}_{\mathsf{L}}^{n} U_p = (UU_{\leq -n})_p = \sum_{j \leq -n} U_{p-j} U_j, \quad \text{and} \qquad \mathrm{N}_{\mathsf{R}}^{n} U_p = (U_{\geq n} U^{\mathsf{R}})_p = \sum_{i \geq n} U_i U_{p-i}.$$

We note that in particular, we have $\mathrm{N}_{\mathsf{R}}^{n+p} U_p = \mathrm{N}_{\mathsf{L}}^{n} U_p$. Through the restriction of the seminorm to these subspaces, we then define a filtered completions of U^{L} and U^{R}, both containing a graded completion of U (see Definition/Lemma A.5.6). Specifically, we set

$$\widehat{U}_d^{\mathsf{L}} := \varprojlim_n \frac{U_{\leq d}^{\mathsf{L}}}{\mathrm{N}_{\mathsf{L}}^{n} U_{\leq d}^{\mathsf{L}}}, \qquad \widehat{U}_d^{\mathsf{R}} := \varprojlim_n \frac{U_{\geq d}^{\mathsf{R}}}{\mathrm{N}_{\mathsf{R}}^{n} U_{\geq d}^{\mathsf{R}}}, \qquad \widehat{U}_d := \varprojlim_n \frac{U_d}{\mathrm{N}_{\mathsf{L}}^{n} U_d} = \varprojlim_n \frac{U_d}{\mathrm{N}_{\mathsf{R}}^{n+d} U_d},$$

And set

$$\widehat{U}^{\mathsf{L}} := \bigcup_d \widehat{U}_d^{\mathsf{L}}, \qquad \widehat{U}^{\mathsf{R}} := \bigcup_d \widehat{U}_d^{\mathsf{R}}, \qquad \widehat{U} := \bigoplus_d \widehat{U}_d.$$

As previously mentioned, it follows from Corollary A.9.10 that this will result in a good triple $(\widehat{U}^{\mathsf{L}}, \widehat{U}, \widehat{U}^{\mathsf{R}})$ of associative algebras with good seminorms.

Finally, we define a graded ideal J of $\widehat{U}$ generated by the Jacobi relations and identification of the identity elements. That is, J is generated by the relations $1_{\widehat{U}} = 1_{[-1]}$ and

$$\sum_{i \geq 0} (-1)^i \binom{\ell}{i} \left(a_{[m+\ell-i]} b_{[n+i]} - (-1)^{\ell} b_{[n+\ell-i]} a_{[m+i]} \right) = \sum_{i \geq 0} \binom{m}{i} (a_{(\ell+i)}(b))_{[m+n-i]}$$

for $\ell, m, n \in \mathbb{Z}$, and $a, b \in V$.

If we let J^{R} and J^{L} be the ideals of $\widehat{U}^{\mathsf{R}}$ and $\widehat{U}^{\mathsf{L}}$ generated by J, and we let $\overline{J}, \overline{J}^{\mathsf{R}}, \overline{J}^{\mathsf{L}}$ be the respective closures (see Definition/Lemma A.5.10), then we find that $(\overline{J}^{\mathsf{L}}, \overline{J}, \overline{J}^{\mathsf{R}})$ form a good triple (of nonunital algebras) by Lemma A.9.5 and Lemma A.9.6. Finally, by Corollary A.9.9, we find that the resulting quotient algebras

$$\mathscr{U}^{\mathsf{L}} = \widehat{U}^{\mathsf{L}} / \overline{J}^{\mathsf{L}}, \quad \mathscr{U} = \widehat{U} / \overline{J}, \quad \mathscr{U}^{\mathsf{R}} = \widehat{U}^{\mathsf{R}} / \overline{J}^{\mathsf{R}},$$

form a good triple of associative algebras with good seminorms (actually norms).

Definition 2.4.3. We call $\mathscr{U}^{\mathsf{L}}, \mathscr{U}^{\mathsf{R}}, \mathscr{U}$ the left, right and finite universal enveloping algebras of V, respectively.

We note that an admissible V-module W—endowed with the discrete topology—naturally has the structure of a $\mathscr{U}$-module (or a $\mathscr{U}^{\mathsf{L}}$-module) such that the actions of these normed (and hence topological) algebras are continuous. That is, such that the multiplication map

$$\mathscr{U} \times W \to W \quad \text{or equivalently,} \quad \mathscr{U}^{\mathsf{L}} \times W \to W$$

is continuous, where W is given the discrete topology, and $\mathscr{U}$ and $\mathscr{U}^{\mathsf{L}}$ are topologized according to their norms.

2.4.4. Relation to the Literature We note that $\mathscr{U}$ coincides with the universal enveloping algebra of V introduced in [FZ92], while we can identify $\mathscr{U}^{\mathsf{L}}$ with the current algebra introduced in [NT05] or with the universal enveloping algebra $\widetilde{U}(V)$ introduced in [FBZ04], with a minor modification (see [Fre07, footnote on p.74]).

2.5. Subalgebras and Subquotient Algebras. We describe here some algebras built from the algebras $\mathscr{U}$, $\mathscr{U}^{\mathsf{L}}$, $\mathscr{U}^{\mathsf{R}}$ which will play a special role. By definition, $\mathscr{U}^{\mathsf{L}}_{\leq -n} \lhd \mathscr{U}^{\mathsf{L}}_{\leq 0}$ and $\mathscr{U}^{\mathsf{R}}_{\geq n} \lhd \mathscr{U}^{\mathsf{R}}_{\geq 0}$ are two-sided ideals when $n > 0$ with

$$\mathscr{U}^{\mathsf{L}}_{\leq 0}/\mathscr{U}^{\mathsf{L}}_{\leq -1} \cong \mathscr{U}_0 \cong \mathscr{U}^{\mathsf{R}}_{\geq 0}/\mathscr{U}^{\mathsf{R}}_{\geq 1},$$

by the fact that these algebras are part of a good triple. We now look more closely at $\mathscr{U}_0$, which forms a subring of $\mathscr{U}$. As our triple of algebras is good, the seminorms on our algebras are almost canonical (Definition A.6.8), and in particular by Definition A.6.8(c), it follows that $\mathrm{N}^n_{\mathsf{L}}\mathscr{U}_0 = \mathrm{N}^n_{\mathsf{R}}\mathscr{U}_0$ for every n, so that there is no ambiguity in denoting these neighborhoods by $\mathrm{N}^r\mathscr{U}_0$. We also see in the same way that $\mathrm{N}^r\mathscr{U}_0$ is a two-sided ideal of $\mathscr{U}_0$.

Associative Zhu algebras $\mathsf{A}_d(V)$ are usually defined as quotients of V [Zhu96, DLM98]. For our goals, it is more practical to define $\mathsf{A}_d(V)$ via an alternative characterization given in [He17].

Definition 2.5.1. The *d-th higher level Zhu algebra* of V is the quotient

$$\mathsf{A}_d(V) = \mathscr{U}_0/\,\mathrm{N}^{d+1}\mathscr{U}_0.$$

For an element $\alpha \in \mathscr{U}_0$, we will write $[\alpha]_d$ for its image in $\mathsf{A}_d(V)$. When V is understood, we will denote $\mathsf{A}_d(V)$ simply by A_d.

2.5.2. Relation to the Literature In [Zhu96] the author defines an associative algebra, now referred to as the (zeroth) Zhu algebra as the quotient of V by an appropriate subspace $O(V)$. In [FZ92, NT05] it is shown that this algebra is isomorphic to an appropriate quotient of the degree zero piece of the universal enveloping algebra of V (or of the current algebra of V). As mentioned in the introduction, higher level Zhu algebras A_d have been introduced in [DLM98] as quotients of V by subspaces $O_d(V)$, and proved to be realized as quotients of the degree zero piece of the universal enveloping algebra of V in [He17]. We notice further that the map that realizes the isomorphism between $V/O_d(V)$ and A_d is explicitly realized by identifying $[a] \in V/O_d(V)$ with the class of the element $a_{\deg(a)-1}$ in $U_0/\,\mathrm{N}^{d+1}U_0$ for every homogeneous element $a \in V$. Other realizations of $\mathsf{A}_d(V)$ have been given, under appropriate assumptions, in [Hua05, vE11, vEH19].

3. Induced Modules and the Mode Transition Algebra

The constructions and results discussed here are true in greater generality, as detailed in Appendix B.1 and in Appendix B.2. For instance, as in Sect. 2, the constructions mentioned in Sect. 3.1 and in Sect. 3.2 hold for a graded vertex algebra which does not necessarily have a conformal structure. The conformal structure is however used in Sect. 3.4.

3.1. Induced Modules. As is the convention, throughout we denote A_0 by A.

Definition 3.1.1. For a left module W_0 over A, we define the *left generalized Verma module* $\Phi^{\mathsf{L}}(W_0)$ as

$$\Phi^{\mathsf{L}}(W_0) = \bigoplus_{p=0}^{\infty} \Phi^{\mathsf{L}}(W_0)_p = \left(\mathscr{U}/\,\mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right) \otimes_{\mathscr{U}_0} W_0 \cong \left(\mathscr{U}^{\mathsf{L}}/\,\mathrm{N}^1_{\mathsf{L}}\mathscr{U}^{\mathsf{L}}\right) \otimes_{\mathscr{U}_0} W_0. \quad (3)$$

For Z_0 a right module over A, we define the *right generalized Verma module* $\Phi^{\mathsf{L}}(W_0)$ as

$$\Phi^{\mathsf{R}}(Z_0) = \bigoplus_{p=0}^{\infty} \Phi^{\mathsf{L}}(Z_0)_{-p} = Z_0 \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right) \cong Z_0 \otimes_{\mathscr{U}_0} \left(\mathscr{U}^{\mathsf{R}} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}^{\mathsf{R}}\right). \quad (4)$$

We note that this is well-defined. In fact, the claimed isomorphisms of (3) and (4) follow from Lemma A.8.1, while the grading is explained in Remark B.1.6.

Moreover, from Lemma A.8.1 we have that $\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U} \cong \mathscr{U}^{\mathsf{L}} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}^{\mathsf{L}}$, so that this quotient can be regarded both as a $(\mathscr{U}, \mathscr{U}_0)$ bimodule and as a $(\mathscr{U}^{\mathsf{L}}, \mathscr{U}_0)$ bimodule. In particular, this shows that the ancillary algebra acts on the left on $\Phi^{\mathsf{L}}(W_0)$.

Analogously to (3), we may also define $\Phi^{\mathsf{L}}_d(W_d)$ for every left A_d-module W_d (see also Definition B.1.2). The functors Φ^{L}_d have a universal property, described in Proposition 3.1.2, and proved more generally in Proposition B.1.4. Namely, given a continuous left $\mathscr{U}$-module W, we define an A_d-module $\Omega_d(W)$ by

$$\Omega_d(W) = \mathrm{rAnn}_W(\mathrm{N}^{d+1}_{\mathsf{L}} U) = \{w \in W \mid (\mathrm{N}^{d+1}_{\mathsf{L}} U)w = 0\},$$

where by Lemma 2.4.2, one has that the right annihilator $\mathrm{rAnn}_W(\mathrm{N}^{d+1}_{\mathsf{L}} U)$ coincides with $\mathrm{rAnn}_W(\mathfrak{L}(V)^{\mathsf{L}}_{\leq -d-1})$. It therefore follows that the definition of $\Omega_d(W)$ agrees with

$$\Omega_d(W) = \{w \in W \mid \mathfrak{L}(V)^{\mathsf{L}}_{-d-1} w = 0\},$$

as originally given in [DLM98].

Proposition 3.1.2. *Let M be a $\mathscr{U}$-module and W_0 an A_d-module. Then there is a natural isomorphism of bifunctors:*

$$\mathrm{Hom}_{\mathsf{A}_d}(W_0, \Omega_d(M)) = \mathrm{Hom}_{\mathscr{U}}(\Phi^{\mathsf{L}}_d(W_0), M).$$

Remark 3.1.3. If W_0 is finite dimensional over $\mathbb{C}$, and if V is C_1-cofinite, then there are a finite number of elements $x^1, x^2, \ldots, x^r \in V$ such that $\Phi^{\mathsf{L}}(W_0)$ is spanned by elements of the form

$$x^1_{(-m_1)} \cdot x^2_{(-m_2)} \cdots x^r_{(-m_r)} \otimes u,$$

for some $u \in W_0$ and positive integers $m_1 \geq m_2 \geq \cdots \geq m_r \geq 1$.

3.2. Mode Transition Algebras and Their Action on Modules. In this section we introduce the *mode transition algebra* $\mathfrak{A}(V)$ associated with a vertex operator algebra V. A general treatment of these algebras is developed in Appendix B, while we will list here the principal consequences of the general theory. We begin by introducing the space underlying $\mathfrak{A}(V)$, often denoted $\mathfrak{A}$ when V is understood.

Definition 3.2.1. Let V be a VOA and $\mathsf{A} = \mathsf{A}_0$ be the Zhu algebra associated to V. We define $\mathfrak{A} = \mathfrak{A}(V)$ to be the vector space

$$\mathfrak{A} = \Phi^{\mathsf{R}}(\Phi^{\mathsf{L}}(\mathsf{A})) = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(\mathsf{A})) = \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right) \otimes_{\mathscr{U}_0} \mathsf{A} \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right).$$

Moreover, using the notation $\mathfrak{A}_{d_1,d_2} = \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right)_{d_1} \otimes_{\mathscr{U}_0} \mathsf{A} \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right)_{d_2}$ we write

$$\mathfrak{A} = \bigoplus_{d_1 \in \mathbb{Z}_{\geq 0}} \bigoplus_{d_2 \in \mathbb{Z}_{\leq 0}} \mathfrak{A}_{d_1,d_2}.$$

The isomorphism described in the following Lemma is crucial to the definition of an algebra structure on $\mathfrak{A}$. We refer to Lemma B.2.1 for its proof.

Lemma 3.2.2. *There is an isomorphism:*

$$\left(\mathscr{U}/\mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right) \otimes_{\mathscr{U}} \left(\mathscr{U}/\mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right) \to A$$
$$\overline{\alpha} \otimes \overline{\beta} \mapsto \alpha \circledast \beta$$

where, for $\alpha, \beta \in \mathscr{U}$ homogeneous, we define $\alpha \circledast \beta$ as:

$$\alpha \circledast \beta = \begin{cases} 0 & \text{if } \deg(\alpha) + \deg(\beta) \neq 0 \\ [\alpha\beta]_0 & \text{if } \deg(\alpha) + \deg(\beta) = 0 \end{cases}$$

and we extend the definition to general products by linearity.

Example 3.2.3. We explicitly describe the element $\alpha \circledast \beta \in \mathsf{A}$ when α and β are homogeneous elements of opposite degrees. Three cases can occur:

- $\deg(\alpha) > 0$. It follows that $\deg(\beta) < 0$ and so $\alpha\beta \in \mathrm{N}^1\mathscr{U}_0$, which gives $\alpha \circledast \beta = 0$.
- $\deg(\alpha) = 0 = \deg(\beta)$. We have that $\alpha \circledast \beta = [\alpha]_0 \cdot [\beta]_0$ since the map $\mathscr{U}_0 \to \mathsf{A}_0$ is a ring homomorphism.
- $\deg(\alpha) < 0$. We first rewrite $\alpha\beta = \beta\alpha + [\alpha, \beta]$. Since $\beta\alpha \in \mathrm{N}^1\mathscr{U}_0$, we have that $\alpha \circledast \beta$ coincide with $[\,[\alpha, \beta]\,]_0$.

Note that if $\alpha, \beta \in \mathfrak{L}(V)^{\mathsf{f}}$—homogeneous and of opposite degree—then $[\alpha, \beta] \in \mathfrak{L}(V)^{\mathsf{f}}_0$, so the above description tells us that $\alpha \circledast \beta$ is computed via the standard map $\mathfrak{L}(V)^{\mathsf{f}}_0 \to \mathsf{A}$ described in [Li94].

Remark 3.2.4. We note that one has that $\mathfrak{A}_{0,0} = \mathsf{A}$. Indeed, by the definitions

$$\mathfrak{A}_{0,0} = \left(\mathscr{U}/\mathrm{N}^1_{\mathsf{L}}\mathscr{U}\right)_0 \otimes_{\mathscr{U}_0} \left(\mathscr{U}_0/\mathrm{N}^1\mathscr{U}_0\right) \otimes_{\mathscr{U}_0} \left(\mathscr{U}/\mathrm{N}^1_{\mathsf{R}}\mathscr{U}\right)_0 \cong \mathsf{A} \otimes_{\mathsf{A}} \mathsf{A} \otimes_{\mathsf{A}} \mathsf{A} \cong \mathsf{A}.$$

We will next simultaneously describe the algebra structure on $\mathfrak{A}$, and the action of this algebra $\mathfrak{A}$ on generalized Verma modules.

Definition 3.2.5. Let W_0 be an A-module. Following Definition B.2.4 we define a map $\star\colon \mathfrak{A} \times \Phi^{\mathsf{L}}(W_0) \to \Phi^{\mathsf{L}}(W_0)$ as follows. For $\mathfrak{a} = u \otimes a \otimes u' \in \mathfrak{A}$ and $\beta \otimes w \in \Phi^{\mathsf{L}}(W_0)$ we set

$$\mathfrak{a} \star (\beta \otimes w) := u \otimes a(u' \circledast \beta)w.$$

By Definition 3.2.5 this map defines an algebra structure on $\mathfrak{A}$ and $\Phi^{\mathsf{L}}(W_0)$ becomes a left $\mathfrak{A}$-module. Moreover the subspace $\mathfrak{A}_d := \mathfrak{A}_{d,-d}$ is closed under multiplication, hence it defines a subalgebra of $\mathfrak{A}$. The following is a special case of Definition B.2.6.

Definition 3.2.6. We call $\mathfrak{A}(V) = \mathfrak{A} = (\mathfrak{A}, +, \star)$ the *mode transition algebra* of V, and $\mathfrak{A}_d = (\mathfrak{A}_d, +, \star)$ the *d-th mode transition algebra* of V.

Remark 3.2.7. More generally, the d-th mode transition algebra $\mathfrak{A}_d$ acts naturally on the degree d part of any weak $\mathbb{N}$-graded module in a way which is compatible with the action of the enveloping algebra (see Proposition B.2.9).

Remark 3.2.8. We observe that the underlying vector space and the algebra structure of $\mathfrak{A}(V)$ does not depend on the existence of a conformal structure on V. Therefore $\mathfrak{A}(V)$ can be defined for every graded vertex algebra V.

We refer to Remark 3.4.6 for an explicit description of the algebra structure of $\mathfrak{A}(V)$ when V is a C_2-cofinite and rational vertex operator algebra. The following assertion is straightforward

Remark 3.2.9. Let W_0 be an A_0-module. Then the action of $\mathfrak{A}_d$ on $\Phi^{\mathsf{L}}(W_0)_d$ factors through the action of A_d described in Definition 3.2.5 via the map μ_d described in Sect. 6 and Lemma B.3.1.

3.2.10. Relation to the Literature In [DJ08] a series of unital associative algebras $\mathsf{A}_{e,d}$, are defined as quotients of V, with $\mathsf{A}_{d,d} \cong \mathsf{A}_d$. By Definition 3.2.5, the $\mathfrak{A}_d$ act on the degree d part of an induced module $W = \Phi^{\mathsf{L}}(W_0)$, as is true for the $\mathsf{A}_{e,d}$, although they differ. In [Hua20], a series of associative algebras $A^d(V)$ is defined which contain the higher level Zhu algebra $\mathsf{A}_d(V)$ as subalgebras, and act on (the sum of) components of an $\mathbb{N}$-graded admissible module up to degree d. In [Hua21], relations are established between bimodules for these associative algebras and (logarithmic) intertwining operators. In [Hua23], a related algebra $A^\infty(V)$ is used to prove modular invariance of (logarithmic) intertwining operators. In Question 9.3.1 we ask about a possible relationship between the algebras $\mathfrak{A}(V)$ and $A^\infty(V)$.

3.3. Strong Unital Action of $\mathfrak{A}_d$ on Modules. The $\mathfrak{A}_0 = \mathsf{A}$ has an identity element given by the image of $\mathbf{1}_{[-1]}$, denoted 1. On the other hand, as we show in Sect. 8, for $d \in \mathbb{Z}_{>0}$, the $\mathfrak{A}_d$ may not admit multiplicative identity elements. However, if there are unities in $\mathfrak{A}_d$ for all $d \in \mathbb{N}$, we have the following results about them.

Definition/Lemma 3.3.1. Let M be an A-module, and assume that for every $d \in \mathbb{N}$ the ring $\mathfrak{A}_d$ is unital, with unity $\mathscr{I}_d \in \mathfrak{A}_d$. We say that $\mathscr{I}_d$ is a strong identity element for every d if one of the following equivalent conditions is verified:

(1) For every $d,n,m \in \mathbb{N}$, for all $u \in \mathfrak{L}(V)_d$, and $\mathfrak{a} \in \mathfrak{A}_{n,-m}$ one has

$$(u \cdot \mathscr{I}_n) \star \mathfrak{a} = u \cdot \mathfrak{a} \qquad \text{and} \qquad \mathfrak{a} \star (\mathscr{I}_m \cdot u) = \mathfrak{a} \cdot u.$$

(2) For every $n, m \in \mathbb{N}$ and for every $\mathfrak{a} \in \mathfrak{A}_{n,-m}$ one has $\mathscr{I}_n \star \mathfrak{a} = \mathfrak{a} = \mathfrak{a} \star \mathscr{I}_m$.
(3) For every $d \in \mathbb{N}$ the homomorphism $\mathfrak{A}_d \to \operatorname{End}(\Phi^{\mathsf{L}}(\mathsf{A})_d)$ is unital;
(4) For every $d \in \mathbb{N}$, the homomorphism $\mathfrak{A}_d \to \operatorname{End}(\Phi^{\mathsf{L}}(\mathsf{A})_d)$ is unital and injective.
(5) For every $d \in \mathbb{N}$ and M an A-module, the homomorphism $\mathfrak{A}_d \to \operatorname{End}(\Phi^{\mathsf{L}}(M)_d)$ is unital.

Proof. We prove these conditions are equivalent. Since (4) implies (3) and (5) implies (3), it will be enough to show the following implications:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (5) \quad \text{and} \quad (3) \Rightarrow (4)$$

(1) ⇒ (2): this follows by taking $d = 0$ and $u = 1$.
(2) ⇒ (1): This follows from Proposition B.2.5.
(2) ⇒ (3): This follows from the identification of $\Phi^{\mathsf{L}}(\mathsf{A})_d$ with $\mathfrak{A}_{d,0}$.

(3) ⇒ (2): By linearity, we can assume that $\mathfrak{a} \in \mathsf{A}_{n,-m}$ is represented by an element of the form $u \otimes a \otimes v$ with $u \in \mathscr{U}_n^{\mathsf{L}}$, $v \in \mathscr{U}_{-m}^{\mathsf{R}}$ and $a \in \mathsf{A}$. Then

$$\mathscr{I}_n \star (u \otimes a \otimes v) = \mathscr{I}_n \star (u \otimes \mathfrak{a} \otimes 1) \cdot (v) = (\mathscr{I}_n \star (u \otimes a)) \otimes v = (u \otimes a \otimes 1) \cdot v = u \otimes a \otimes v$$

where (3) ensures that the third equality holds.

(3) ⇒ (5): By linearity we can assume that an element of $\Phi^{\mathsf{L}}(M)_d$ is given by $u \otimes m$ for $u \in \mathscr{U}_d^{\mathsf{L}}$ and $m \in M$. Hence we obtain $\mathscr{I}_d \star (u \otimes m) = (\mathscr{I}_d \star u) \otimes m = u \otimes m$.

(3) ⇒ (4): By Definition 3.2.5 the action of an element $\mathfrak{a} \in \mathfrak{A}_d$ via $\star$ on $\mathfrak{A}_d \subseteq \mathfrak{A} = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(\mathsf{A})) = \Phi^{\mathsf{L}}(\mathsf{A}) \otimes_{\mathscr{U}_0} \left(\mathscr{U} / \mathrm{N}_{\mathsf{R}}^1 \mathscr{U}\right)$ is determined by the action of $\mathfrak{a}$ on $\Phi^{\mathsf{L}}(\mathsf{A})$. In particular, as the former is injective when we have an identity element, the latter must be injective in this case as well. □

3.4. Relation to the Functor Φ. To define the mode transition algebra $\mathfrak{A}$, we used the map $\Phi^{\mathsf{L}}\Phi^{\mathsf{R}} = \Phi^{\mathsf{L}}\Phi^{\mathsf{R}}$, which we can interpret as a functor from the category of A-bimodules to the category of $\mathscr{U}$-bimodules. Its properties in a more general framework are described in Proposition B.2.5.

We now show that this functor agrees with the functor denoted Φ in [DGK22, Definition 2.2] which assigns to an A-bimodule M the $(\mathscr{U}^{\mathsf{L}})^{\otimes 2}$-module

$$\Phi(M) := (\mathscr{U}^{\mathsf{L}} \otimes \mathscr{U}^{\mathsf{L}}) \underset{\mathscr{U}^{\mathsf{L}}_{\leq 0} \otimes \mathscr{U}^{\mathsf{L}}_{\leq 0}}{\widehat{\otimes}^{\mathsf{f}}} M,$$

where $\mathscr{U}^{\mathsf{L}}_{\leq 0} \otimes \mathscr{U}^{\mathsf{L}}_{\leq 0}$ acts on M as follows:

$$(u \otimes v)(m) = \begin{cases} u \cdot m \cdot \theta(v) & \text{if } u, v \in \mathscr{U}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Here θ is the natural involution of $\mathscr{U}_0$, from e.g. [DGK22, Eq.(7)]), and which we describe briefly in Section Sect. 3.4.1. In Lemma 3.4.5 we describe the relation between functors Φ^{L} and Φ^{R} to Φ.

3.4.1. The Involutions, Left and Right Actions As we explain here, there is an anti-Lie algebra isomorphism θ used to transport the universal enveloping algebra, considered as an object that acts on modules on the left (denoted here by $\mathscr{U}^{\mathsf{L}}$), to an analogous completion $\mathscr{U}^{\mathsf{R}}$ that acts on modules on the right.

The map $\theta \colon V \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!) \to V \otimes_{\mathbb{C}} \mathbb{C}(\!(t^{-1})\!)$ is defined, for $a \in V$ homogeneous, by

$$a \otimes \sum_{i \geq N} c_i t^i \mapsto (-1)^{\deg(a)} \sum_{j \geq 0}^{\deg(a)} \left(\frac{1}{j!} \left(L_1^j(a) \right) \otimes \sum_{i \geq N} c_i t^{2\deg(a) - i - j - 2} \right), \tag{5}$$

and extended linearly.

The map θ is related to the involution $\gamma = (-1)^{L_0} e^{L_1} \colon V \to V$ defined, for $a \in V$ homogeneous by

$$a \mapsto (-1)^{\deg a} \sum_{i \geq 0} \frac{1}{i!} L_1^i(a), \tag{6}$$

and extended by linearity. To state the relation succinctly, for every homogeneous $a \in V$ we set

$$J_n(a) := a_{[\deg(a)-1+n]}. \tag{7}$$

This notation, used in [NT05], has the property that $\deg(J_n(v)) = -n$, so that the degree of such an element is easily read.

Lemma 3.4.2. *For $a \in V$, homogeneous, $\theta(J_n(a)) = J_{-n}(\gamma(a))$.*

Proof. This follows by combining (5) and (6) and using linearity. □

Lemma 3.4.3. *The map θ induces a Lie algebra anti-isomorphism $\mathfrak{L}(V)^{\mathsf{L}} \to \mathfrak{L}(V)^{\mathsf{R}}$, which restricts to a Lie algebra involution on $\mathfrak{L}(V)^{\mathsf{f}}$, such that $\theta(\mathfrak{L}(V)^{\mathsf{L}}_{\leq d}) = \mathfrak{L}(V)^{\mathsf{R}}_{\geq -d}$ and $\theta(\mathfrak{L}(V)^{\mathsf{f}}_{d}) = \mathfrak{L}(V)^{\mathsf{f}}_{-d}$.*

Proof. One can check that this restricts to an endomorphism of $V \otimes \mathbb{C}[t, t^{-1}]$, which by [NT05, Proposition 4.1.1], defines a Lie algebra involution of $\mathfrak{L}(V)^{\mathsf{f}}$, that is, $\theta([\ell_1, \ell_2]) = -[\theta(\ell_1), \theta(\ell_2)]$, and $\theta^2 = \mathrm{id}$. Moreover, it is easy to verify that $\theta(\mathfrak{L}(V)^{\mathsf{f}}_{d}) \subset \mathfrak{L}(V)^{\mathsf{f}}_{-d}$. As the Lie algebras $\mathfrak{L}(V)^{\mathsf{L}}$, $\mathfrak{L}(V)^{\mathsf{R}}$ carry exhaustive and separated split filtrations by the graded subalgebra $\mathfrak{L}(V)^{\mathsf{f}}$, they are naturally equipped with norms via Remark A.4.5 – that is, by declaring that elements of large positive degree are large in $\mathfrak{L}(V)^{\mathsf{L}}$ and small in $\mathfrak{L}(V)^{\mathsf{R}}$. With this definition, it follows that θ is continuous, and as noted in Remark A.4.8, that the multiplication on the Lie algebras is continuous. Finally, the fact that $\mathfrak{L}(V)^{\mathsf{f}}$ induces a splitting of the filtrations, it follows that $\mathfrak{L}(V)^{\mathsf{f}}$ is simultaneously dense in $\mathfrak{L}(V)^{\mathsf{L}}$ and $\mathfrak{L}(V)^{\mathsf{R}}$. Consequently, we see by continuity that θ induces an anti-homomorphism from $\mathfrak{L}(V)^{\mathsf{L}}$ to $\mathfrak{L}(V)^{\mathsf{R}}$, which is an anti-isomorphism as $\theta^2 = \mathrm{id}$. □

If $R = (R, +, \cdot)$ is a ring, denote by R^{op} its opposite ring, that is $R^{\mathrm{op}} = (R, +, *)$ where $a * b := b \cdot a$. Similarly, if $(L, [\ ,\ ])$ is a Lie algebra, we denote by L^{op} its opposite Lie algebra, where $[a, b]_{L^{\mathrm{op}}} := [b, a]_L$.

Lemma 3.4.4. *For $U(\mathfrak{L}(V)^{\mathsf{R}})$ the universal enveloping algebra of $\mathfrak{L}(V)^{\mathsf{R}}$,*

$$\theta\colon U(\mathfrak{L}(V)^{\mathsf{L}}) \to U(\mathfrak{L}(V)^{\mathsf{R}})^{\mathrm{op}}$$

is an isomorphism of rings.

Proof. We have established in Lemma 3.4.3 that $\theta\colon \mathfrak{L}(V)^{\mathsf{L}} \to \mathfrak{L}(V)^{\mathsf{R}}$ is an anti-isomorphism of Lie algebras, so that $\theta\colon \mathfrak{L}(V)^{\mathsf{L}} \to (\mathfrak{L}(V)^{\mathsf{R}})^{\mathrm{op}}$ is a Lie-algebra isomorphism. Moreover, as $U((\mathfrak{L}(V)^{\mathsf{R}})^{\mathrm{op}}) = U(\mathfrak{L}(V)^{\mathsf{R}})^{\mathrm{op}}$, it follows that θ induces an isomorphism between $U(\mathfrak{L}(V)^{\mathsf{L}})$ and $U(\mathfrak{L}(V)^{\mathsf{R}})^{\mathrm{op}}$, as wanted. Here we note that $\theta(\alpha \cdot \beta) = \theta(\beta) \cdot \theta(\alpha)$ for every $\alpha, \beta \in \mathfrak{L}(V)^{\mathsf{L}}$ (and where $\cdot$ is the usual product in the $U(\mathfrak{L}(V)^{\mathsf{L}})$ and $U(\mathfrak{L}(V)^{\mathsf{R}})$). □

In particular θ is an isomorphism between $U(\mathfrak{L}(V)^{\mathsf{f}})$ and $U(\mathfrak{L}(V)^{\mathsf{f}})^{\mathrm{op}}$.

Lemma 3.4.5. *Let B be an associative ring, W^1 an (A, B)-bimodule and W^2 a (B, A)-bimodule. Then we have a natural identification*

$$\Phi^{\mathsf{L}}(W^1) \otimes_B \Phi^{\mathsf{R}}(W^2) \cong \Phi(W^1 \otimes_B W^2)$$

of $(\mathscr{U}^{\mathsf{L}}, \mathscr{U}^{\mathsf{R}})$-bimodules. In particular we have $\mathfrak{A} = \Phi(\mathsf{A})$.

Proof. We first note that there is a natural equivalence of categories between left $(\mathscr{U}^{\mathsf{L}})^{\otimes 2}$-modules and $(\mathscr{U}^{\mathsf{L}}, (\mathscr{U}^{\mathsf{L}})^{\mathrm{op}})$-modules. Moreover, as described in Lemma 3.4.4, the involution θ provides an identification $\mathscr{U}^{\mathsf{R}} \cong (\mathscr{U}^{\mathsf{L}})^{\mathrm{op}}$. It follows that the map $\Phi^{\mathsf{L}}(W^1) \otimes_B \Phi^{\mathsf{R}}(W^2) \to \Phi(W^1 \otimes_B W^2)$ induced by

$$(u \otimes w_1) \otimes (w_2 \otimes v) \mapsto (u \otimes \theta(v)) \otimes (w_1 \otimes w_2)$$

for all $u \in \mathscr{U}^{\mathsf{L}}$, $v \in \mathscr{U}^{\mathsf{R}}$ and $w_i \in W_i$ is indeed an isomorphism.

Remark 3.4.6. Let V be a rational VOA. Then the mode transition algebras $\mathfrak{A}_d$ have strong identity elements. To see this, we note that the rationality of V implies that A is finite and semi-simple, and thus has a bimodule decomposition

$$\mathsf{A} \cong \prod_{k=1}^{m} I^k = \prod_{k=1}^{m} W_0^k \otimes (W_0^k)^\vee,$$

where $\{1, \dots, m\}$ is the set indexing the isomorphism classes of simple V-modules. Equivalently, W_0^k runs over all isomorphism classes of simple left A-modules (so that $W_0^\vee$ is the corresponding dual right A-module). By Lemma 3.4.5 we therefore have:

$$\mathfrak{A} = \Phi(\mathsf{A}) = \Phi\left(\prod_{k=1}^{m} W_0^k \otimes (W_0^k)^\vee\right) = \prod_{k=1}^{m} \Phi^{\mathsf{L}}(W_0^k) \otimes_{\mathbb{C}} \Phi^{\mathsf{R}}(W_0^k)^\vee.$$

Note that since V is rational, the functor Φ^{L} takes simple A-modules to simple V-modules. This follows from the fact that Φ^{L} preserves indecomposable modules [DGK22, Lemma 1.2], and that for a rational VOA, indecomposable modules are simple. Hence for W_0 a simple A-module, $W := \Phi^{\mathsf{L}}(W_0)$ must be simple as well. Similarly, we have that also $\Phi^{\mathsf{R}}(W_0^\vee)$ must be a simple V-module. We give some details about this: the involution θ identifies left and right A-modules allowing us to consider $(W_0^k)^\vee$ as a left A-module which we denote as ${}^{\theta}(W_0^k)^\vee$. From this perspective, and the above discussions relating Φ^{L}, Φ^{R} and θ, we thus have a natural identification

$$\Phi^{\mathsf{L}}({}^{\theta}(W_0^k)^\vee) = \Phi^{\mathsf{R}}((W_0^k)^\vee) \quad \text{and} \quad \Phi^{\mathsf{L}}({}^{\theta}(W_0^k)^\vee)_d = \Phi^{\mathsf{R}}((W_0^k)^\vee)_{-d},$$

for every $d \in \mathbb{N}$. Since W_0^k was simple, also ${}^{\theta}(W_0^k)^\vee$ is simple, and thus we conclude that $\Phi^{\mathsf{R}}((W_0^k)^\vee)$ is simple as well.

We now show that $\Phi^{\mathsf{R}}((W_0^k)^\vee)$ is isomorphic to the contragredient module $(W^k)'$ of W^k. The contragredient module $(W^k)'$ has zero part $(W^k)'_0 = {}^{\theta}(W_0^k)^\vee$ inducing a natural map $\Phi^{\mathsf{R}}((W_0^k)^\vee) = \Phi^{\mathsf{L}}({}^{\theta}(W_0^k)^\vee) \to (W^k)'$. As the contragredient of a simple V-module is simple, it follows that this map is an isomorphism. In particular one has $\Phi^{\mathsf{R}}((W_0^k)^\vee)_{-d} = (W_d^k)^\vee$ and thus

$$\mathfrak{A} = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \prod_{k=1}^{m} W_d^k \otimes_{\mathbb{C}} (W_d^k)^\vee. \tag{8}$$

Using this decomposition, the $\star$-product is induced, by linearity, from

$$(a_{W^k} \otimes \varphi_{W_d^k}) \star (b_{W_e^j} \otimes \psi_{W^j}) = \begin{cases} \varphi_{W_d^k}(b_{W_e^j})(a_{W^k} \otimes \psi_{W^j}) & \text{if } k = j \text{ and } e = d \\ 0 & \text{otherwise,} \end{cases}$$

where $\varphi_{W_d^k}\colon W_d^k \to \mathbb{C}$ and $b_{W_e^j} \in W_e^j$. Note that, by rationality, the spaces W_d^k are necessarily finite dimensional [Zhu96, Definition 1.2.4]. It then follows that, for all $d \in \mathbb{Z}_{\geq 0}$, we have an isomorphism of rings

$$\mathfrak{A}_d \cong \prod_{i=1}^{m} \mathrm{End}_{\mathbb{C}}(W_d^k)$$

and thus $\mathbf{1}_d := \prod_{i=1}^m \mathrm{Id}_{W_d^k}$ is its strong identity element. We remark that [NT05, Proposition 7.2.1] states, without a detailed proof and using a different notation, that (8) holds, under the assumption that V is not only rational, but also C_2-cofinite. From our argument it is apparent how C_2-cofinitess is not needed.

4. Smoothing, Limits, and Coinvariants

In Sect. 4.1 we describe the sheaf of coinvariants on schemes S parametrizing families of pointed and coordinatized curves in general terms, while in Sect. 4.2, we explain what we mean by sheaves defined over a scheme $S = \mathrm{Spec}(R)$, where R is a ring complete with respect to some ideal I. In Sect. 4.3, we describe the setup for considering coinvariants on smoothings of nodal curves, establishing some results needed for our geometric applications. In particular, in Sect. 4.4, for the proof of Proposition 5.1.2, we explicitly describe the sheaf $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ of Chiral Lie algebras.

Throughout this section, V is a VOA of CFT-type and every V-module is assumed to be admissible (as already specified in Notation 1.1.3).

4.1. Coinvariants. Let S be a scheme and let $\mathscr{W}$ be a quasi-coherent sheaf of $\mathscr{O}_S$-modules. Let $\mathscr{L}$ be a quasi-coherent sheaf of Lie algebras on S acting on $\mathscr{W}$. We define the *sheaf of coinvariants* $[\mathscr{W}]_{\mathscr{L}}$ on S as the cokernel

$$\mathscr{L} \otimes_{\mathscr{O}_S} \mathscr{W} \to \mathscr{W} \to [\mathscr{W}]_{\mathscr{L}} \to 0.$$

For future use, it will be helpful to note that the formation of the sheaves of coinvariants commutes with base change.

Lemma 4.1.1. *Let $\mathscr{L}$ be a quasi-coherent sheaf of Lie algebras on a scheme S acting on a quasi-coherent sheaf $\mathscr{W}$. For any morphism $S' \to S$ between two such schemes, we have $([\mathscr{W}]_{\mathscr{L}})_{S'} \cong [\mathscr{W}_{S'}]_{\mathscr{L}_{S'}}$.*

Proof. This follows from right exactness of pullback of quasi-coherent sheaves (equivalently right exactness of tensor). □

Remark 4.1.2. Let $\pi : \mathscr{C} \to S$ be a projective curve, with n distinct smooth sections $P_\bullet\colon S \to \mathscr{C}$ and formal coordinates $t_\bullet$ at $P_\bullet$. Assume further that $\mathscr{C}\setminus \sqcup P_\bullet(S) \to S$ is affine. This assumption is possible by Propagation of Vacua [Cod19, Thm 3.6] (see also [DGT24, Theorem 4.3.1]). Denote by $W^\bullet = W^1 \otimes \cdots \otimes W^n$ the tensor product of an n-tuple of V-modules and let $\mathscr{W} := W^\bullet \otimes \mathcal{O}_S$. The sheaf of Chiral Lie algebras $\mathscr{L} := \mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$, originally defined in this context for families of stable curves with singularities in [DGT21, DGT24], is explicitly described with more details here in Sect. 4.4. The sheaf of coinvariants $[\mathscr{W}]_{\mathscr{L}}$ defined above will also be denoted $[W^\bullet]_{(\mathscr{C},P_\bullet,t_\bullet)}$. While quasi-coherent [DGT21], for V C_2-cofinite, or if generated in degree 1, this sheaf is coherent [DGK22, DG23].

4.2. Completions. In the setting of Sect. 4.1, we consider coinvariants over $S = \mathrm{Spec}(R)$, where R is a ring that is complete with respect to some ideal I. For $k \in \mathbb{Z}_{\geq 0}$, setting $S_k = \mathrm{Spec}(R_k) = \mathrm{Spec}(R/I^{k+1})$, pullbacks $\mathscr{L}_k$ and $\mathscr{W}_k$ of $\mathscr{L}$ and $\mathscr{W}$ to S_k respectively, we work with coinvariants $[\mathscr{W}_k]_{\mathscr{L}_k}$ for any $k \in \mathbb{Z}_{\geq 0}$. Due to quasicoherence, each of these can be thought of as a module over R_k, with maps $[\mathscr{W}_{k+1}]_{\mathscr{L}_{k+1}} \to [\mathscr{W}_k]_{\mathscr{L}_k}$.

Definition 4.2.1. In the above situation, we define the *formal coinvariants*, denoted $\widehat{[\mathscr{W}]_{\mathscr{L}}}$ to be the I-adically complete R-module

$$\widehat{[\mathscr{W}]_{\mathscr{L}}} = \varprojlim [\mathscr{W}_k]_{\mathscr{L}_k}.$$

From the definitions if follows that formal coinvariants can also be expressed as follows:

Proposition 4.2.2. *We have an identification*

$$\widehat{[\mathscr{W}]_{\mathscr{L}}} = \mathrm{coker}\left[\varprojlim \pi_* \mathscr{L}_k \otimes_{\mathscr{O}_{S_k}} \mathscr{W}_k \longrightarrow \varprojlim \mathscr{W}_k\right]$$

Proposition 4.2.3. *Suppose $[\mathscr{W}]_{\mathscr{L}}$ is finitely generated over an I-adically complete Noetherian ring R. Then the natural map $[\mathscr{W}]_{\mathscr{L}} \to \widehat{[\mathscr{W}]_{\mathscr{L}}}$ is an isomorphism.*

Proof. Consider the exact sequence of R-modules (omitting the π_* from the notation, and identifying the quasicoherent sheaves with the corresponding R-modules):

$$\mathscr{L} \otimes_R \mathscr{W} \longrightarrow \mathscr{W} \longrightarrow [\mathscr{W}]_{\mathscr{L}} \longrightarrow 0.$$

Tensoring with R/I^k (or geometrically base-changing along $S_k \to S$) is a right exact operation, hence it yields an exact sequence

$$\mathscr{L}_k \otimes_{R_n} \mathscr{W}_k \longrightarrow \mathscr{W}_k \longrightarrow ([\mathscr{W}]_{\mathscr{L}})_{R_k} \longrightarrow 0,$$

which shows that we can identify $[\mathscr{W}_k]_{\mathscr{L}_k} = ([\mathscr{W}]_{\mathscr{L}})_{R_k}$. In particular, the composition

$$[\mathscr{W}]_{\mathscr{L}} \longrightarrow \widehat{[\mathscr{W}]_{\mathscr{L}}} \longrightarrow [\mathscr{W}_k]_{\mathscr{L}_k}$$

coincides with the surjection $[\mathscr{W}]_{\mathscr{L}} \to [\mathscr{W}]_{\mathscr{L}} \otimes_R R/I^k$. Since $[\mathscr{W}]_{\mathscr{L}}$ is finitely generated over a complete Noetherian ring, it is I-adically complete by [Sta23, Tag 00MA(3)]. Therefore we can identify $[\mathscr{W}]_{\mathscr{L}} = \varprojlim [\mathscr{W}]_{\mathscr{L}} \otimes_R R/I^k = \varprojlim [\mathscr{W}_k]_{\mathscr{L}_k} = \widehat{[\mathscr{W}]_{\mathscr{L}}}$, giving the desired isomorphism. □

4.3. Smoothing Setup. In order to introduce the smoothing property for V, we will recall the notion of a smoothing of a nodal curve, and set a small amount of notation used throughout. Let $R = \mathbb{C}[\![q]\!]$ and write $S = \mathrm{Spec}(R)$. Let $\mathscr{C}_0$ be a projective curve over $\mathbb{C}$ with at least one node Q, smooth and distinct points $P_\bullet = (P_1, \ldots, P_n)$ such that $\mathscr{C}_0 \backslash P_\bullet$ is affine, and formal coordinates $t_\bullet = (t_1, \ldots, t_n)$ at $P_\bullet$. Let $\eta \colon \widetilde{\mathscr{C}_0} \to \mathscr{C}_0$ be the partial normalization of $\mathscr{C}_0$ at Q, which is naturally pointed by $Q_\pm := \eta^{-1}(Q)$. We also suppose we have chosen formal coordinates at $Q_\pm$ and we call them $s_\pm$.

The choice of our formal coordinates $s_\pm$ determine a smoothing family $(\mathscr{C}, P_\bullet, t_\bullet)$ over S, with the central fiber given by $(\mathscr{C}_0, P_\bullet, t_\bullet)$. Let $(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)$ denote the trivial extension $\widetilde{\mathscr{C}_0} \times S$ with its corresponding markings. We will now discuss the relationship between coinvariants for $(\mathscr{C}, P_\bullet, t_\bullet)$ and $(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)$.

Let $W^1, \dots, W^n$ be an n-tuple of V-modules, which we interpret as smooth $\mathscr{U}$-modules for $\mathscr{U}$ the universal enveloping algebra of V (defined in Sect. 2), and denote by $W^\bullet$ their tensor product. As is described above in Remark 4.1.2, we may also consider the sheaf of coinvariants $[W^\bullet]_{(\mathscr{C},P_\bullet,t_\bullet)}$.

As mentioned in the introduction, there is a map $\alpha_0\colon W^\bullet \to W^\bullet \otimes \Phi(\mathsf{A})$ which induces a map between coinvariants

$$[\alpha_0]\colon [W^\bullet]_{(\mathscr{C}_0,P_\bullet,t_\bullet)} \longrightarrow [W^\bullet \otimes \Phi(\mathsf{A})]_{\left(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm\right)}.$$

Moreover, if V is C_1-cofinite, then we will show in Lemma 4.4.4 that $[\alpha_0]$ is an isomorphism. We recall that $\Phi(\mathsf{A}) = \mathfrak{A}$, so we will generally use the notation $\mathfrak{A}$ below.

The following result, which is a consequence of Proposition 4.2.3, allows us to describe coinvariants over $\widetilde{\mathscr{C}}$ whenever they are finite dimensional. The assumptions of the following result are satisfied when V is C_2-cofinite, for all V-modules $W^\bullet$, and also more generally (by [DGK22]).

Corollary 4.3.1. *Assume that the sheaf* $[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm)}$ *is coherent over* S. *Then one has identifications*

$$\begin{aligned}
&[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{\left(\widetilde{\mathscr{C}},P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm\right)}\\
&\cong [W^\bullet \otimes \mathfrak{A}]_{\left(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm\right)}[\![q]\!]\\
&\cong [W^\bullet \otimes \mathfrak{A}]_{\left(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm\right)} \otimes_{\mathbb{C}} \mathbb{C}[\![q]\!].
\end{aligned}$$

Proof. The second isomorphism holds because the coherence assumption implies that $[W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm)}$ is a finite dimensional vector space. To prove the first isomorphism, we consider, for $R = \mathbb{C}[\![q]\!]$, the R-module and R-Lie algebra

$$\mathscr{W} = (W^\bullet \otimes \mathfrak{A})[\![q]\!], \quad \text{and} \quad \mathscr{L} = \mathcal{L}_{\widetilde{\mathscr{C}}\setminus\{P_\bullet\sqcup Q_\pm\}}(V).$$

Since $[\mathscr{W}]_{\mathscr{L}}$ is a finite dimensional R-module, for $R_k = \mathbb{C}[\![q]\!]/q^{k+1}$, and ($S_k = \mathrm{Spec}(R_k)$), one can show by Proposition 4.2.3, that

$$[\mathscr{W}]_{\mathscr{L}} = \varprojlim \left([\mathscr{W} \otimes_R R_k]_{\mathscr{L}\otimes_R R_k}\right). \tag{9}$$

Note further that $\mathscr{W} \otimes_R R_k = (W^\bullet \otimes \mathfrak{A}) \otimes_{\mathbb{C}} R_k$, and similarly,

$$\mathscr{L} \otimes_R R_k = \mathcal{L}_{\widetilde{\mathscr{C}}_0\setminus\{P_\bullet\sqcup Q_\pm\}}(V) \otimes_{\mathbb{C}} R_k.$$

Using this, together with Proposition 4.2.2, we deduce that (9) is isomorphic to

$$\varprojlim \left([W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm)} \otimes_{\mathbb{C}} R_k\right)$$

which is indeed $[W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0,P_\bullet\sqcup Q_\pm,t_\bullet\sqcup s_\pm)}[\![q]\!]$, as was asserted. □

Remark 4.3.2. Corollary 4.3.1 implies that, up to some assumptions of coherence, the sheaf of coinvariants associated with $W^\bullet \otimes \mathfrak{A}$ over $\widetilde{\mathscr{C}}_0$ deforms trivially to the sheaf of coinvariants over the trivial deformation $\widetilde{\mathscr{C}}$ of $\widetilde{\mathscr{C}}_0$. Consequently, the target of the induced map $[\alpha]$, which extends the map $[\alpha_0]$ is therefore identified with the sheaf of coinvariants associated with $\widetilde{\mathscr{C}}$ (and not only with a completion thereof).

We conclude this section with some criteria to show coherence of sheaves of coinvariants over S. Throughout we will use the notation $R_k = \mathbb{C}[\![q]\!]/q^{k+1}$ and $S_k = \mathrm{Spec}(R_k)$ for every $k \in \mathbb{N}$.

Lemma 4.3.3. *For M any module over R_k, let $m_1, \ldots, m_r \in M$ be elements whose images generate $M \otimes_{R_k} R_0$. Then the elements $m_1, \ldots, m_r$ also generate M.*

Proof. We induct on k, the case $k = 0$ being automatic. For the induction step, suppose $m \in M$ and consider the R_{k-1} module $\overline{M} = M \otimes_{R_k} R_{k-1}$. By the induction hypothesis, the elements $m_1, \ldots, m_r$ generate $\overline{M}$. Therefore we can find $a_1, \ldots a_r \in R_k$ so that

$$m' = m - \sum a_i m_i \in M,$$

maps to 0 in $\overline{M}$.

Now consider the submodule $M' = q^k M \subset M$. As $q^k M$ is exactly the kernel of the map $M \to \overline{M} = M \otimes_{R_k} R_{k-1}$ we find that $m' \in M'$, and therefore we can write $m' = q^k x$ for some $x \in M$. Writing $x = \sum b_i m_i \pmod q$, we find $x - \sum b_i m_i = qy$ for some $y \in M$. But now we have

$$m = \left(\sum a_i m_i\right) + m' = \left(\sum a_i m_i\right) + q^k \left(\left(\sum b_i m_i\right) + qy\right) = \sum (a_i + q^k b_i) m_i,$$

as desired. □

Proposition 4.3.4. *If $[W^\bullet]_{(\mathscr{C}_0, P_\bullet, t_\bullet)}$ is a finite dimensional vector space, then both*

$$[W^\bullet[\![q]\!]]_{(\mathscr{C}, P_\bullet, t_\bullet)} \quad \textit{and} \quad [(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{\left(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm\right)}$$

are coherent.

Proof. For every $k \in \mathbb{N}$ and for every scheme X over S, denote the pullback of S to S_k by X_k. Define

$$M_k := [W^\bullet_{R_k}]_{(\mathscr{C}_k, P_\bullet, t_\bullet)} \quad \text{and} \quad \widetilde{M}_k := [(W^\bullet \otimes \mathfrak{A})_{R_k}]_{\left(\widetilde{\mathscr{C}}_k, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm\right)}.$$

Let us first show that M_k and $\widetilde{M}_k$ are coherent. As we are considering modules over the Noetherian ring R_k, we only need to show that they are finitely generated. But by Lemma 4.3.3, for this it suffices to show that $\widetilde{M}_0$ and M_0 are finitely generated. This holds because by assumption M_0 is finitely generated and $\alpha_0 \colon M_0 \to \widetilde{M}_0$ is an isomorphism by [DGT24].

For simplicity, denote

$$M = [W^\bullet[\![q]\!]]_{(\mathscr{C}, P_\bullet, t_\bullet)} \quad \text{and} \quad \widetilde{M} = [(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{\left(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm\right)}.$$

By Lemma 4.1.1, it follows that $M_k = M \otimes_R R_k$ and $\widetilde{M}_k = \widetilde{M} \otimes_R R_k$. Consequently the natural maps $\widetilde{M}_k \to \widetilde{M}_{k-1}$ and $M_k \to M_{k-1}$ are surjective. It follows therefore from [Sta23, Lemma 087W] that M and $\widetilde{M}$ will be finitely generated over R whenever M_k and $\widetilde{M}_k$ are finitely generated over R_k for every k. This is what we have just shown and so M and $\widetilde{M}$ are coherent. □

4.4. The Sheaf of Chiral Lie Algebras. The sheaf of Chiral Lie algebras $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ can be identified with a quotient of the space of sections of the sheaf $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$ on the affine open set $\mathscr{C} \setminus P_\bullet \subset \mathscr{C}$ (see [DGT21,DGT24]). Here, for later use in the proof of Proposition 5.1.2, in order to describe the action of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$, we explicitly describe the sheaf $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$, where $\mathcal{V}_{\mathscr{C}}$ is the contracted product $(V \otimes_{\mathbb{C}} \mathcal{O}_C) \times_{\mathcal{A}ut\mathcal{O}} \mathcal{A}ut_{\mathscr{C}}$ (see Remark 4.4.2).

For this, suppose we are given a relative curve $\mathscr{C}$, projective over $(S = \operatorname{Spec} \mathbb{C}[\![q]\!])$, with closed fiber $\mathscr{C}_0$ (cut out by the ideal generated by q), and an $(n+1)$-tuple of distinct closed points $P_0, \ldots, P_n \in \mathscr{C}_0$ with affine complement $\mathscr{C}_0 \setminus P_\bullet = \mathscr{C}_0 \setminus \bigcup_i P_i$. Let $B = \mathcal{O}_{\mathscr{C}}(\mathscr{C}_0 \setminus P_\bullet)$ denote those rational functions on $\mathscr{C}$ which are regular at every scheme-theoretic point of $\mathscr{C}_0 \setminus P_\bullet$ and let $\widehat{B}$ denote its q-adic completion. By [Pri00, Theorem 3.4], coherent sheaves on $\mathscr{C}$ may be described by specifying coherent sheaves M_U on $U = \operatorname{Spec} \widehat{B}$, coherent sheaves M_i on $(D_i := \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},P_i})$ for each i, together with "gluing data on the overlaps."

The overlaps in this case are described as the formal completions $D_i^\times$ of the fiber products $(\operatorname{Spec} \widehat{B} \times_{\mathscr{C}} \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},P_i})$, and the gluing data is a choice of an isomorphism $(M_i)_{D_i^\times} \cong (M_U)_{D_i^\times}$. More concretely, the $D_i^\times$ can be described as follows. In a given complete local ring $\widehat{\mathcal{O}}_{\mathscr{C},P_i}$, the ideal generated by q which describes the closed fiber will factor into a product of of primes $\wp_{i,j}$. For each of these we can consider the localization and completion at the prime. We find that $D_i^\times$ is the disjoint union of the formal spectra of the rings $\left((\widehat{\mathcal{O}}_{\mathscr{C},P_i})_{\wp_{i,j}}\right)^{\widehat{}\,\wp_{i,j}}$. In particular, a coherent sheaf over $D_i^\times$ is the data of a finitely generated module over the Noetherian ring $\left((\widehat{\mathcal{O}}_{\mathscr{C},P_i})_{\wp_{i,j}}\right)^{\widehat{}\,\wp_{i,j}}$.

In our case, we consider a semistable family of curves $\mathscr{C}/S$, such that $\mathscr{C}$ is a regular scheme and the closed fiber is reduced. We focus our attention on an isolated node Q, and choose points $P_\bullet$ with $Q = P_0$ and with $\mathscr{C}_0 \setminus P_\bullet$ smooth. We then find that in $\widehat{\mathcal{O}}_{\mathscr{C},Q}$, the complete (regular) local ring at Q, we may factor $q = s_+ s_-$. Consequently, we may write $\widehat{\mathcal{O}}_{\mathscr{C},Q} \cong \mathbb{C}[\![s_+, s_-]\!]$. That is, we have

$$\widehat{\mathcal{O}}_{\mathscr{C},Q} \cong \mathbb{C}[\![s_+, s_-, q]\!]/(s_+ s_- - q) \cong \mathbb{C}[\![s_+, s_-]\!].$$

In this case, if we let $\wp_+$ be the prime generated by s_- and $\wp_-$ be the prime generated by s_+ (in $\widehat{\mathcal{O}}_{\mathscr{C},Q}$), then we find

$$\left((\widehat{\mathcal{O}}_{\mathscr{C},P_i})_{\wp_\pm}\right)^{\widehat{}\,\wp_\pm} = \mathbb{C}(\!(s_\pm)\!)[\![q]\!].$$

As $\mathcal{V}_{\mathscr{C}}$ (and similarly $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$) is a limit of coherent sheaves $(\mathcal{V}_{\mathscr{C}})_{\leq k}$, we may use the above procedure to describe it.

We choose U so that the torsor $\mathcal{A}ut_{\mathscr{C}/S}$ is trivial over $(\operatorname{Spec} \widehat{B})$ via the choice of a function $s \in \widehat{B}$ such that ds is a free generator of $\omega_{\mathscr{C}/S}(\operatorname{Spec} \widehat{B})$ as an $\widehat{B}$-module. In other words, s is a coordinate on U. In particular, sections of $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$ on $\operatorname{Spec} \widehat{B}$ can be described as the $\widehat{B}$ module:

$$\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\operatorname{Spec} \widehat{B}) = \bigoplus_{k \in \mathbb{N}} V_k \otimes_{\mathbb{C}} \widehat{B}\, (d/ds)^{k-1}. \tag{10}$$

Remark 4.4.1. It is important to note that these expressions are not intrinsic to $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$ as a sheaf on $\mathscr{C}$, but rather depend on a choice of parameter s. Different choices give different identifications which correspond to inhomogeneous isomorphisms between the direct sums, but which do preserve the filtrations $(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S})_{\leq k}$.

Similarly, on $D_Q = Spec(\widehat{\mathcal{O}}_{\mathscr{C},\mathscr{Q}})$, either s_+ or s_- can be used to define a trivialization of the torsor $\mathcal{A}ut_{\mathscr{C}}$, this time corresponding to the two possible choices of generators ds_+/s_+ or ds_-/s_- of $\omega_{\mathscr{C}/S}$. These choices allow us to give the following expressions for the sections of our sheaf on D_Q as a $\widehat{\mathcal{O}}_{\mathscr{C},Q}$-module:

$$\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(D_Q) = \bigoplus_{k\in\mathbb{N}} V_k \otimes_{\mathbb{C}} \mathbb{C}[\![s_+, s_-]\!] s_{\pm}^{k-1}(d/ds_{\pm})^{k-1}. \tag{11}$$

In particular, we may express a section σ on D_Q with respect to either the trivialization given by s_+ or by s_-. Since $\gamma(s_+) = s_-$, the trivializations of $\mathcal{A}ut_{\mathscr{C}}$ associated to the coordinates s_+ and s_- (regarded as sections of the torsor) are related by the order 2 element $(-1)^{L_0}e^{L_1} \in \mathcal{A}ut\mathcal{O}$, which acts on V via the involution γ described in Eq. (6). Hence, we can write sections of the contracted product $(V \otimes_{\mathbb{C}} \mathcal{O}_{\mathscr{C}}) \times_{\mathcal{A}ut\mathcal{O}} \mathcal{A}ut_{\mathscr{C}}$ over D_Q as

$$(v \otimes f, s_+) = (v \otimes f, \gamma s_-) \sim (\gamma(v) \otimes f, s_-)\,,$$

for $f \in \mathcal{O}_{\mathscr{C}}$. Choosing $v \in V_\ell$, the element of (11) which in the s_+ trivialization is represented by

$$\sum_{i,j\geq 0} v \otimes x_{i,j} s_+^i s_-^j s_+^{\ell-1}(d/ds_+)^{\ell-1},$$

is represented with respect to the s_- trivialization as

$$\sum_{i,j\geq 0}\sum_{m=0}^{\ell} \frac{1}{m!} L_1^m v \otimes x_{i,j} s_+^i s_-^j s_-^{\ell-m-1}(d/ds_-)^{\ell-m-1}.$$

More generally, one should consider a sum of such terms for various values of ℓ.

Finally we consider the sheaf $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$ on $D_{\pm}^{\times} = \mathrm{Spec}(\mathbb{C}(\!(s_{\pm})\!)[\![q]\!])$. In $D_{\pm}^{\times}$, as in D_Q, we may use the functions $s_{\pm}$ to trivialize our torsor. Consequently we have:

$$\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(D_{\pm}) = \bigoplus_{k\in\mathbb{N}} V_k \otimes_{\mathbb{C}} \mathbb{C}(\!(s_{\pm})\!)[\![q]\!]\, s_{\pm}^{k-1}(d/ds_{\pm})^{k-1}. \tag{12}$$

Without loss of generality the trivializing coordinate s on U maps to our previously chosen trivializing coordinate s_+ in $D_+^{\times}$. That is, the map $i_+\colon D_+^{\times} \hookrightarrow U$ corresponds to maps of rings

$$\widehat{B} \to \mathbb{C}(\!(s_+)\!)[\![q]\!],\ s \mapsto s_+. \tag{13}$$

Although it is unnecessary here, to map s to both s_+ and s_- simultaneously, one could work étale locally.

For notational convenience, it is useful to consider the action of $\mathcal{A}ut\mathcal{O}$ as on $\mathfrak{L}(V)_0^{\mathsf{f}}$, the degree 0 part of the ancillary algebra and to recall the notation (7). For $\rho \in \mathcal{A}ut\mathcal{O}$ and a homogeneous element $a \in V$, we have $\rho J_0(a) = J_0(\rho a)$. Further, when we use a coordinate s to trivialize our torsor $\mathcal{A}ut_{\mathscr{C}}$, we will identify the expression $a_{[\deg(a)-1+k]}$ with the element $J_k(a) \in \mathfrak{L}(V)_{-k}^{\mathsf{f}}$. Finally, we simplify notation further by omitting the factors of the form d/ds from our presentations.

Given $\sigma_U \in \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\mathrm{Spec}\,\widehat{B})$ we write $(\sigma_U)_{\pm}$ for its restriction to $D_{\pm}^{\times}$. Using the notation above, following the explicit expressions of (10) and (12), we find that if

$$\sigma_U = \sum_{\ell=0}^{k} v_\ell \otimes f_\ell,$$

then writing f_+ for the expansion (restriction) of the regular function f to $\mathbb{C}(\!(s_+)\!)[\![q]\!]$, we have (as the coordinates are compatible)

$$(\sigma_U)_+ = \sum_{\ell=0}^{k} v_\ell \otimes (f_\ell)_+ = \sum v_\ell \otimes (g_\ell)_+ s_+^{\ell-1}.$$

On the other hand, if $\sigma_Q \in \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(D_Q)$, is written as $\sum_{\ell=0}^{k} \sum_{i,j\geq 0} v_\ell \otimes x_{i,j}^{\ell} s_+^{i+\ell-1} s_-^{j}$. If the section σ_Q, so represented, is to be compatible and glue together with the section σ_U above, we find that

$$\sum_{\ell=0}^{k} \sum_{i,j\geq 0} J_0(v_\ell) x_{i,j}^{\ell} s_+^{i} s_-^{j} = \sum_{i,j,\ell} J_0(v_\ell) x_{i,j}^{\ell} s_+^{i-j} q^{j} = \sum_{i,j,\ell} J_{i-j}(v_\ell) x_{i,j}^{\ell} q^{j} \tag{14}$$

must represent the expression for σ_U restricted to $D_+^{\times}$.

To express σ_U restricted to $D_-^{\times}$, following (5), we will make use of the anti-isomorphism $\theta\colon \mathfrak{L}(V)^{\mathsf{L}} \to \mathfrak{L}(V)^{\mathsf{R}}$ described in (5) and related to γ via Lemma 3.4.2. We then conclude that σ_U restricted to $D_-^{\times}$ is given by the expression

$$\begin{aligned}\sum_{\ell=0}^{k} \sum_{i,j\geq 0} \gamma(J_0(v_\ell)) x_{i,j}^{\ell} s_+^{i} s_-^{j} &= \sum_{i,j,\ell} J_0(\gamma(v_k)) x_{i,j}^{\ell} s_-^{j-i} q^{i} \\ &= \sum_{i,j,\ell} J_{j-i}(\gamma(v_k)) x_{i,j}^{\ell} q^{i} = \sum_{i,j,\ell} \theta(J_{i-j}(v_\ell)) x_{i,j}^{\ell} q^{i}.\end{aligned}$$

Remark 4.4.2. Through the above description, we have that the sheaf $\mathcal{V}_{\mathscr{C}}$ discussed at length in [DGT24] agrees, even on the boundary of $\overline{\mathcal{M}}_{g,n}$, with the sheaf $\mathscr{V}_{\mathscr{C}}$ described in [DGT21].

We conclude with two lemmas which will be useful in our applications in the next section.

Lemma 4.4.3. *Let $\mathscr{C}$ be a family of curves over S, possibly with nodal singularities. Consider a collection of sections $P_1, \ldots, P_n$ such that $\mathscr{C} \setminus P_\bullet = U \subset \mathscr{C}$ is affine, and let $Q_1, \ldots Q_k \subset \mathscr{C}$ be a finite collection of distinct closed points in U (possibly including nodes). Let $D_{Q_i} = \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q_i}$ be the complete local ring at Q_i with maximal ideal $\widehat{\mathfrak{m}}_{\mathscr{C},Q_i}$. Then for any $\ell \geq 0$ and any invertible sheaf of $\mathcal{O}_{\mathscr{C}}$-modules $\mathcal{L}$, the natural map*

$$\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)(U) \to \bigoplus_i \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q_i} / \left(\widehat{\mathfrak{m}}_{\mathscr{C},Q_i}\right)^{\ell})$$

is surjective.

Proof. As $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L} = \bigcup_k \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)_{\leq k}$, it suffices to show that the map

$$\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)_{\leq k}(U) \to \bigoplus_i \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)_{\leq k}(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q_i} / \left(\widehat{\mathfrak{m}}_{\mathscr{C},Q_i}\right)^{\ell})$$

is surjective for all k. Since the sheaf $\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \mathcal{L}\right)_{\leq k}$ is free of finite rank over $\mathcal{O}_{\mathscr{C}}$, then this holds true. Indeed, for any coherent sheaf of modules M on $\mathscr{C}$, the natural map $M(U) \to \bigoplus_i M(\operatorname{Spec} \mathcal{O}_{\mathscr{C}}(U)/\mathfrak{m}_{\mathscr{C},Q_i}(U)^\ell) = \bigoplus_i M(U) \otimes_{\mathcal{O}_{\mathscr{C}}(U)} \mathcal{O}_{\mathscr{C}}(U)/\mathfrak{m}_{\mathscr{C},Q_i}(U)^\ell$ is seen to be surjective, using the fact that $\mathcal{O}_{\mathscr{C}}(U) \to \bigoplus_i \mathcal{O}_{\mathscr{C}}(U)/\mathfrak{m}_{\mathscr{C},Q_i}(U)^\ell$ is surjective by the Chinese Remainder Theorem, and that tensoring with M is right exact. □

Lemma 4.4.4. *As in Sect. 4.3, let $\mathscr{C}_0$ be a projective curve over $\mathbb{C}$ with at least one node Q, smooth and distinct points $P_\bullet = (P_1, \ldots, P_n)$ such that $\mathscr{C}_0 \setminus P_\bullet$ is affine, and formal coordinates $t_\bullet = (t_1, \ldots, t_n)$ at $P_\bullet$. Let $\eta\colon \widetilde{\mathscr{C}}_0 \to \mathscr{C}_0$ be the partial normalization of $\mathscr{C}_0$ at Q, pointed by $Q_\pm := \eta^{-1}(Q)$, and choose formal coordinates $s_\pm$ at $Q_\pm$. Let $W^1, \ldots, W^n$ be an n-tuple of V-modules. Then the map $\alpha_0\colon W^\bullet \to W^\bullet \otimes \mathfrak{A}$ defined by $\alpha_0(w) = w \otimes 1$ induces a map between the vector spaces of coinvariants:*

$$[\alpha_0]\colon [W^\bullet]_{(\mathscr{C}_0, P_\bullet, t_\bullet)} \longrightarrow [W^\bullet \otimes \mathfrak{A}]_{\left(\widetilde{\mathscr{C}}_0, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm\right)},$$

which is an isomorphism in case V is C_1-cofinite.

Proof. Suppose $\mathscr{C}_0$ has m nodes in total (including Q) and let $\widetilde{\mathscr{C}}_0{}'$ be the (full) normalization of $\mathscr{C}_0$. Following [DGK22, Remark 3.4] we find we have maps

$$W^\bullet \xrightarrow{\alpha_0} W^\bullet \otimes \mathfrak{A} \xrightarrow{\alpha_0''} W^\bullet \otimes \mathfrak{A}^{\otimes m} \quad \text{and we set } \alpha_0' = \alpha_0''\alpha_0.$$

These induce corresponding maps $[\alpha_0]$, $[\alpha_0']$, $[\alpha_0'']$ on the respective coinvariants such that $[\alpha_0']$ is an isomorphism. It follows that $[\alpha_0]$ is injective and therefore remains only to show that it is also surjective.

For surjectivity, we follow the spirit of the proof of [DGT24, Prop. 6.2.1]. We may represent an element of $\mathfrak{A}$ as given by an expression $a^1_{[n_1]} \cdots a^k_{[n_k]} \otimes 1 \otimes b^1_{[m_1]} \cdots b^r_{[m_r]}$. For simplicity of notation, let us write $a = a^1_{[n_1]} \cdots a^k_{[n_k]}$, $a' = a^2_{[n_2]} \cdots a^k_{[n_k]}$ and $b = b^1_{[m_1]} \cdots b^r_{[m_r]}$. We will show that all elements of the form $[w \otimes (a \otimes 1 \otimes b)]$ are in the image of α_0 by induction on $k - m$, the base case $k - m = 0$ being true by construction (note b has nonpositive degree by definition). For the induction step, let us suppose that $k > 0$ (the case $m < 0$ being similar), and let d_+' be the degree of a' and d_- the degree of b. Without loss of generality, we may assume $\deg(a^1_{[n_1]}) \geq \cdots \geq \deg(a^k_{[n_k]}) \geq 0$. By Lemma 4.4.3, setting $\mathcal{L} = \omega_{\mathscr{C}_0}(n_1 Q_+ + N Q_-)$ for $N > d_- - \deg(a^1)$, we may find a section $\sigma = a^1 \otimes f$,

$$\sigma \in \left(\mathcal{V}_{\widetilde{\mathscr{C}}_0} \otimes_{\mathcal{O}_{\mathscr{C}_0}} \omega_{\mathscr{C}_0}(n_1 Q_+ + (d_- - 1) Q_-)\right)(\widetilde{\mathscr{C}}_0 \backslash P_\bullet) \subset \left(\mathcal{V}_{\widetilde{\mathscr{C}}_0} \otimes_{\mathcal{O}_{\mathscr{C}_0}} \omega_{\mathscr{C}_0}\right)(\widetilde{\mathscr{C}}_0 \backslash P_\bullet \sqcup Q_\pm)$$

such that the image $\sigma^{\mathsf{L}}_{Q_+}$ of σ in $\left(\mathcal{V}_{\widetilde{\mathscr{C}}_0} \otimes_{\mathcal{O}_{\mathscr{C}_0}} \omega_{\mathscr{C}_0}\right)(\widehat{\mathcal{O}}_{\widetilde{\mathscr{C}}_0, Q_+}) \cong \mathfrak{L}(V)^{\mathsf{L}}$ has the form $a^1_{[n_1]} + \widetilde{a}$ where $\deg(\widetilde{a}) < -d_+'$. By construction, $\sigma^{\mathsf{L}}_{Q_-}$ has degree $< d_-$ and consequently $\sigma^{\mathsf{R}}_{Q_+}$ has degree $> -d_-$. So we find $\sigma^{\mathsf{L}}_{Q_+}(a' \otimes 1 \otimes b) = a \otimes 1 \otimes b$ and $(a \otimes 1 \otimes b)\sigma^{\mathsf{R}}_{Q_-} = 0$. This tells us

$$\sigma \cdot \left(w \otimes (a' \otimes 1 \otimes b)\right) = (\sigma w) \otimes (a' \otimes 1 \otimes b) + w \otimes (a \otimes 1 \otimes b)$$

yielding $[w \otimes (a \otimes 1 \otimes b)] = -[(\sigma w) \otimes (a' \otimes 1 \otimes b)]$, completing the induction step. □

5. Smoothing via Strong Identity Elements

Here we prove Theorem 5.0.3, which relates the smoothing property of V, a VOA of CFT-type, described here in Definition 5.0.1, to the existence of strong identity elements in $\mathfrak{A}$. Theorem 5.0.3 relies crucially on Proposition 5.1.2. These results are proved in Sect. 5.1. Geometric consequences regarding coinvariants are given in Sect. 5.2.

Throughout this section we will use the notation introduced in Sect. 4.3, considering two families of marked, parametrized curves $(\mathscr{C}, P_\bullet, t_\bullet)$ and $(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)$ over the base scheme $S = \mathrm{Spec}(\mathbb{C}[\![q]\!])$. As usual $\mathfrak{A}_0 = \mathsf{A}$ and every V-module is assumed to be admissible.

Definition 5.0.1. Given a family $(\mathscr{C}, P_\bullet, t_\bullet)$, and collection of V-modules $W^1, \ldots, W^n$, an element $\mathscr{I} = \sum_{d\geq 0} \mathscr{I}_d q^d \in \mathfrak{A}[\![q]\!]$ defines a *smoothing map* for $W^\bullet$ over $(\mathscr{C}, P_\bullet, t_\bullet)$, if $\mathscr{I}_0 = 1 \in \mathfrak{A}_0$, and the map $W^\bullet \to W^\bullet \otimes \mathfrak{A}[\![q]\!]$, $w \mapsto w \otimes \mathscr{I}$ extends by linearity and q-adic continuity to an $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$-module homomorphism $\alpha\colon W^\bullet[\![q]\!] \longrightarrow (W^\bullet \otimes \mathfrak{A})[\![q]\!]$. We say that $\mathscr{I} = \sum_{d\geq 0} \mathscr{I}_d q^d \in \mathfrak{A}[\![q]\!]$ defines a *smoothing map* for V, if it defines a smoothing map for all V-modules $W^\bullet$, over all families $(\mathscr{C}, P_\bullet, t_\bullet)$.

Definition 5.0.2. Smoothing holds for $W^\bullet$ over the family $(\mathscr{C}, P_\bullet, t_\bullet)$, if there is an element $\mathscr{I} = \sum_{d\geq 0} \mathscr{I}_d q^d \in \mathfrak{A}[\![q]\!]$ giving a smoothing map for $W^\bullet$ over $(\mathscr{C}, P_\bullet, t_\bullet)$. *$V$ satisfies smoothing* if smoothing holds for all $W^\bullet$, over all families $(\mathscr{C}, P_\bullet, t_\bullet)$.

Theorem 5.0.3. *Let V be a VOA. Then the algebras $\mathfrak{A}_d$ admit strong identity elements for all $d \in \mathbb{N}$ if and only if V satisfies smoothing.*

5.1. Proof of Theorem 5.0.3. Following the idea of Definition/Lemma 3.3.1(2), we make the following definition:

Definition 5.1.1. We say that a sequence $(\mathscr{I}_d)_{d\in\mathbb{N}}$, with $\mathscr{I}_d \in \mathfrak{A}$ satisfies the *strong identity element equations* if for every homogeneous $a \in V$, and $n \in \mathbb{Z}$ such that $n \leq d$, we have

$$J_n(a)\mathscr{I}_d = \mathscr{I}_{d-n} J_n(a). \tag{15}$$

In Definition 5.1.1 there is no assumption on the (bi-)degrees of the elements $\mathscr{I}_d \in \mathfrak{A}$. However, if $\mathscr{I}_d \in \mathfrak{A}_d$ is an identity element for each d, then by Definition/Lemma 3.3.1 they satisfy the strong identity element equations if and only if they are strong identity elements. We refer to Lemma 5.1.5 for a generalization of this (see also Lemma 5.1.4).

Proposition 5.1.2. *Let V be a VOA and let $\mathscr{I}_d \in \mathfrak{A}$ for $d \in \mathbb{N}$ and let $W^\bullet$ be a nonzero tensor product of admissible V-modules. Then $\mathscr{I} = \sum \mathscr{I}_d q^d$ defines a smoothing map for $W^\bullet$ over $(\mathscr{C}, P_\bullet, t_\bullet)$ if and only if the sequence $(\mathscr{I}_d)$ satisfies the strong identity element equations* (15).

Proof. The map $\alpha\colon W^\bullet[\![q]\!] \to (W^\bullet \otimes \mathfrak{A})[\![q]\!]$ is a map of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$-modules if and only if, for every $\sigma \in \mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ and for every $u \in W^\bullet$, one has $\alpha(\sigma(u)) = \sigma(\alpha(u))$. Here, the left hand side equals $(\sigma \cdot u) \otimes \mathscr{I}$. To describe the right hand side, as is explained in the beginning of Sect. 4.4, we recall that elements of the Lie algebra $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ are represented by sections of the sheaf $\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}$ over the affine open set $\mathscr{C} \setminus P_\bullet$. Consequently we can understand the right hand side in terms of the maps

$$\begin{aligned}\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\mathscr{C} \setminus P_\bullet) \to \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(D_\pm^\times) &\cong V \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!)[\![q]\!] \\ \sigma \mapsto \sigma_\pm^{\mathsf{L}}.&\end{aligned}$$

We let $\sigma_-^{\mathsf{R}} = \theta(\sigma_-^{\mathsf{L}}) \in \mathbb{C}(\!(t^{-1})\!)[\![q]\!]$. We then have

$$\sigma(u \otimes \mathscr{I}) = \sigma(u) \otimes \mathscr{I} + u \otimes (\sigma_+^{\mathsf{L}} \otimes 1 + 1 \otimes \sigma_-^{\mathsf{R}})(\mathscr{I}).$$

Choosing $u \neq 0$, it follows that α is a map of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$-modules if and only if

$$\sigma \cdot \mathscr{I} = \left(\sigma_+^{\mathsf{L}} \otimes 1 + 1 \otimes \sigma_-^{\mathsf{R}}\right) \cdot \mathscr{I} = 0. \tag{16}$$

We now reframe this in the language developed towards the end of Sect. 4.4. For a section $\sigma \in \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)_{\leq k} (\mathscr{C}\setminus P_\bullet)$, writing $s_+ s_- = q$ on $\widehat{\mathcal{O}}_{\mathscr{C},Q}$, we may write (in terms of the local trivializations of Sect. 4.4)

$$\sigma|_{D_Q} = \sum_{\ell=0}^{k} \sum_{i,j \geq 0} J_0(v_\ell)\, x_{i,j}^{\ell}\, s_+^i s_i^j$$

and for this section σ we have

$$\sigma_+^{\mathsf{L}} = \sum_{\ell=0}^{k} \sum_{i,j \geq 0} J_{i-j}(v_\ell) x_{i,j}^{\ell} q^j \quad \text{and} \quad \sigma_-^{\mathsf{R}} = \sum_{\ell=0}^{k} \sum_{i,j \geq 0} J_{i-j}(v_\ell) x_{i,j}^{\ell} q^i.$$

Putting this together with (16), we find that smoothing holds if and only if for all σ as above (and for all k), we have

$$\sum_{\ell=0}^{k} \sum_{i,j,d \geq 0} x_{ij} \left(J_{i-j}(v_\ell) \cdot \mathscr{I}_d\, q^{d+j} - \mathscr{I}_d \cdot J_{i-j}(v_\ell)\, q^{d+i}\right) = 0.$$

This in turn holds if and only if each coefficient of q^m is zero, translating to the statement

$$\sum_{\ell=0}^{k} \sum_{0 \leq i,j \leq m} x_{ij} \left(J_{i-j}(v_\ell) \cdot \mathscr{I}_{m-j} - \mathscr{I}_{m-i} \cdot J_{i-j}(v_\ell)\right) = 0, \tag{17}$$

for every $m \geq 0$.

We note that the systems of equations

$$J_n(v_\ell)\mathscr{I}_d = \mathscr{I}_{d-n} J_n(v_\ell), \text{ with } n \leq d, \text{ and } d \in \mathbb{N}, v_\ell \in V_\ell, \ell \in \mathbb{N},$$

and

$$J_{i-j}(v_\ell)\cdot\mathscr{I}_{m-j} - \mathscr{I}_{m-i}\cdot J_{i-j}(v_\ell) = 0, \text{ with } 0 \leq i, j \leq m, \text{ and } m \in \mathbb{N}, v_\ell \in V_\ell, \ell \in \mathbb{N},$$

are equivalent after a change of variables. We then have showed that, if $(\mathscr{I}_d)_{d\in\mathbb{N}}$ satisfies the strong identity element equations, it follows that $\mathscr{I}$ defines a smoothing map. It remains to show the converse, namely that if (17) holds for every σ, then the strong identity element equations hold.

We do this by the following strategy: we will show that for every $0 \leq i_0, j_0 \leq m$, $m \in \mathbb{N}$ and $v'_{\ell_0} \in V_{\ell_0}$, we may find a section $\sigma \in \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right) (\mathscr{C}\setminus P_\bullet)$ so that the expansion of σ at Q has the form

$$\sigma|_{D_Q} = \sum_{\ell=0}^{k} \sum_{i,j \geq 0} J_0(v_\ell)\, x_{i,j}^{\ell}\, s_+^i s_i^j = J_0(v'_{\ell_0}) s_+^{i_0} s_-^{j_0} + \sum_{\ell=0}^{k} \sum_{i,j \geq m} J_0(v_\ell)\, x_{i,j}^{\ell}\, s_+^i s_i^j. \tag{18}$$

That is, we argue that the coefficients $x_{i,j}$ in the terms in (18) of degree less than m are only nonzero in the case $i = i_0$, $j = j_0$, and in this case $x_{i_0,j_0} = 1$. For such a section σ, (17) simply becomes $J_{i_0-j_0}(v_\ell) \cdot \mathscr{I}_{m-j_0} - \mathscr{I}_{m-i_0} \cdot J_{i_0-j_0}(v_\ell) = 0$, which, as has been noted, is equivalent to the strong identity element equations once we run this argument for all i_0, j_0 and m. For this final step, we note that by Lemma 4.4.3 we have a surjective map

$$\begin{aligned}\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\mathscr{C} \setminus P_\bullet) &\to \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)_{\leq k}(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q}/\left(\widehat{\mathfrak{m}}_{\mathscr{C},Q}\right)^{2m}) \\ &= \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)_{\leq k}(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q}) \otimes_{\widehat{\mathcal{O}}_{\mathscr{C},Q}} \widehat{\mathcal{O}}_{\mathscr{C},Q}/(\widehat{\mathfrak{m}}_{\mathscr{C},Q})^{2m}\end{aligned}$$

for every $m \geq 0$. Hence there exists $\sigma \in \left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\mathscr{C} \setminus P_\bullet)$ whose image in $\left(\mathcal{V}_{\mathscr{C}} \otimes_{\mathcal{O}_{\mathscr{C}}} \omega_{\mathscr{C}/S}\right)(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{C},Q})$ is congruent modulo $(\widehat{\mathfrak{m}}_{\mathscr{C},Q})^{2m} = (s_+, s_-)^{2m}$ to $J_0(v'_{\ell_0}) s_+^{i_0} s_-^{j_0}$. It follows (18) holds for this σ as desired and the proof is complete. $\square$

We note that already Proposition 5.1.2 shows that the smoothing property never depends on modules or specific families of curves:

Corollary 5.1.3. *Smoothing holds for nonzero $W^\bullet$ over a family $(\mathscr{C}, P_\bullet, t_\bullet)$ if and only if V satisfies smoothing.*

Proof. If smoothing holds for $W^\bullet$ over a family $(\mathscr{C}, P_\bullet, t_\bullet)$, then by Proposition 5.1.2, the sequence $(\mathscr{I}_d)_{d\in\mathbb{Z}}$ satisfies the strong identity element equations. But then, invoking again Proposition 5.1.2, we deduce that this sequence defines a smoothing map for any choice of modules and family of curves. $\square$

In what follows, for an element $b \in \mathfrak{A} = \bigoplus_{i,j} \mathfrak{A}_{i,j}$, we write $b_{i,j} \in \mathfrak{A}_{i,j}$ for the corresponding homogeneous component of b.

Lemma 5.1.4. *For V a VOA, if the sequence $(\mathscr{I}_d)_{d\in\mathbb{N}}$ with $\mathscr{I}_d \in \mathfrak{A}$ satisfies the strong identity element equations (15), then so does the sequence $(\mathscr{I}'_d)_{d\in\mathbb{N}}$ where $\mathscr{I}'_d := (\mathscr{I}_d)_{d,-d}$.*

Proof. Suppose we have a sequence $(\mathscr{I}_d)$ satisfying the strong identity element equations. When we equate terms of like degree in the expression

$$J_n(a) \cdot \mathscr{I}_d = \mathscr{I}_{d-n} \cdot J_n(a),$$

we obtain

$$J_n(a) \cdot (\mathscr{I}_d)_{i+n,j} = (\mathscr{I}_d)_{i,j-n} \cdot J_n(a)$$

for every i, j. In particular, for $\mathscr{I}'_d = (\mathscr{I}_d)_{d,-d}$, we find that the strong identity element equations (15) hold for the sequence $(\mathscr{I}'_d)_{d\in\mathbb{N}}$, as was claimed. $\square$

In what follows we will use the following equalities, which are a direct consequence of Proposition B.2.5. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. Then for every $u, w \in \mathscr{U}$ we have

$$u \cdot (\mathfrak{a} \star \mathfrak{b}) = (u \cdot \mathfrak{a}) \star \mathfrak{b} \qquad \text{and} \qquad (\mathfrak{a} \star \mathfrak{b}) \cdot w = \mathfrak{a} \star (\mathfrak{b} \cdot w). \tag{19}$$

Lemma 5.1.5. *Suppose we have a collection of elements $\mathscr{I}_d \in \mathfrak{A}_d$ for each $d \geq 0$, with $\mathscr{I}_0 = 1 \in \mathfrak{A}_0$. Then, $\mathscr{I}_d$ is a strong identity element in $\mathfrak{A}_d \subset \mathfrak{A}$, for all $d \in \mathbb{N}$, if and only if the sequence $(\mathscr{I}_d)_{d\in\mathbb{N}}$ satisfies the strong identity element equations (15).*

Proof. Definition/Lemma 3.3.1(2) with $\mathfrak{a} = J_n(v)$ implies that strong identity elements satisfy the strong identity element equations (15), so we are left to prove the converse statement. To show that $\mathscr{I}_d$ is a strong identity element for each d, it suffices to show that $\mathscr{I}_d$ acts as the identity element on $\mathfrak{A}_{d,e}$ for every $e \in \mathbb{Z}$. That is, for every $\mathfrak{a} \in \mathfrak{A}_{0,e}$, and $n_1 \le \cdots \le n_r < 0$ with $\sum n_i = -d$, we need to show

$$\mathscr{I}_d \star \big(J_{n_1}(v_1) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big) = J_{n_1}(v_1) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}.$$

We argue by induction on r. The base case $r = 0$ holds since by assumptions $\mathscr{I}_0 = 1 \in \mathfrak{A}_0 = \mathsf{A}$, hence $\mathscr{I}_0 \star \mathfrak{a} = 1 \cdot \mathfrak{a} = \mathfrak{a}$. For the inductive step, we write:

$$\begin{aligned} \mathscr{I}_d \star \big(J_{n_1}(v_1) \cdot J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big) &= \mathscr{I}_d \star \Big(\big(J_{n_1}(v_1)\big)\,\big(J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big)\Big) \\ \text{(19)} \quad &= \big(\mathscr{I}_d \cdot J_{n_1}(v_1)\big) \star \big(J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big) \\ \text{(15)} \quad &= \big(J_{n_1}(v_1) \cdot \mathscr{I}_{d+n_1}\big) \star \big(J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big) \\ \text{(19)} \quad &= J_{n_1}(v_1) \cdot \Big(\mathscr{I}_{d+n_1} \star \big(J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}\big)\Big) \\ \textit{(by induction)} \quad &= J_{n_1}(v_1) J_{n_2}(v_2) \cdots J_{n_r}(v_r) \cdot \mathfrak{a}, \end{aligned}$$

where the last identity holds by induction. □

We may now complete the proof of Theorem 5.0.3.

Proof of Theorem 5.0.3. Suppose the algebras $\mathfrak{A}_d$ admit strong identity elements. Writing $\mathscr{I}_d$ for these unities, we can apply Lemma 5.1.5 to deduce that the sequence $(\mathscr{I}_d)_{d\in\mathbb{N}}$ satisfies the strong identity element equations and therefore, by Proposition 5.1.2, the element $\mathscr{I} = \sum \mathscr{I}_d q^d$ defines a smoothing map for any family of marked curves and choice of modules $W^\bullet$. Hence V satisfies smoothing.

Conversely, if V satisfies smoothing, there exists $\mathscr{I} = \sum \mathscr{I}_d q^d$ which defines a smoothing map for any family of marked curves and choice of modules $W^\bullet$, then by Proposition 5.1.2, the sequence $(\mathscr{I}_d)_{d\in\mathbb{N}}$ satisfies the strong identity element equations. Using Lemma 5.1.4 we may find a new sequence $(\mathscr{I}'_d)_{d\in\mathbb{N}}$ with $\mathscr{I}'_d \in \mathfrak{A}_d$ which also satisfy the strong identity element equations. It follows from Lemma 5.1.5 that the elements $\mathscr{I}'_d$ are strong identity elements. □

5.2. Geometric Results. We describe in this section some statements about coinvariants, most of which are implications of Theorem 5.0.3.

Corollary 5.2.1. *For any VOA V, let $W^\bullet$ be V-modules such that the sheaf $[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}}, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}$ is coherent over S. Assume that $\mathfrak{A}_d$ admits a strong identity element $\mathscr{I}_d$ for every $d \in \mathbb{N}$. Set $\mathscr{I} = \sum_{d\ge 0} \mathscr{I}_d q^d$, and let $\alpha\colon W^\bullet[\![q]\!] \to (W^\bullet \otimes \mathfrak{A})[\![q]\!]$ be the map induced by $w \mapsto w \otimes \mathscr{I}$ (see Definition 5.0.1). Then the diagram*

$$\begin{array}{ccc} [W^\bullet[\![q]\!]]_{(\mathscr{C}, P_\bullet, t_\bullet)} & \xrightarrow{\quad [\alpha] \quad} & [W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}[\![q]\!] \\ \downarrow & & \downarrow \\ [W^\bullet]_{(\mathscr{C}_0, P_\bullet, t_\bullet)} & \xrightarrow{\quad [\alpha_0] \quad} & [W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0, P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)} \end{array}$$

commutes, where $\alpha_0\colon W^\bullet \to W \otimes \mathfrak{A}$ is given by $w \mapsto w \otimes \mathscr{I}_0$.

Proof. The vertical maps are given by imposing the condition $q = 0$, and are surjective. After the identification of $\mathfrak{A}$ with $\Phi(\mathsf{A})$ provided in Lemma 3.4.5, we see that the map $[\alpha_0]$ is well-defined as in [DGK22, Proposition 3.3].

By the proof of Theorem 5.0.3 we deduce that the map α is a map of $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$-modules and since $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V) \subset \mathcal{L}_{\widetilde{\mathscr{C}}\setminus(P_\bullet \sqcup Q_\pm)}(V)$, this induces a map of coinvariants

$$[W^\bullet[\![q]\!]]_{(\mathscr{C},P_\bullet,t_\bullet)} \longrightarrow [(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}$$

whose reduction modulo q is indeed $[\alpha_0]$. Finally, we use Corollary 4.3.1 to identify

$$[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)} \cong [W^\bullet \otimes \mathfrak{A}]_{(\widetilde{\mathscr{C}}_0,P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}[\![q]\!]$$

which concludes the proof. □

To state the following consequence, we recall that sheaves of coinvariants $\mathbb{V}(V; W^\bullet)$ are attached to coordinatized curves $(C, P_\bullet, t_\bullet)$ such that $C \setminus P_\bullet$ is affine. By Propagation of vacua [Cod19, DGT24], we may drop the latter condition, so that $\mathbb{V}(V; W^\bullet)$ can be considered a sheaf on $\widehat{\overline{\mathcal{M}}}_{g,n}$, the stack of stable coordinatized curves. Depending on V and on $W^\bullet$, this further descends to a sheaf over $\overline{\mathcal{M}}_{g,n}$. To formulate our next result it is convenient to introduce the following notation.

Definition 5.2.2. Let V be a VOA. We say that V has *coherent coinvariants* if for every family of stable and pointed coordinatized curves $(C, P_\bullet, t_\bullet)$, and modules $W^\bullet$, the sheaf of coinvariants $[W^\bullet]_{(C,P_\bullet,t_\bullet)}$ is coherent.

Definition 5.2.3. Let V be a VOA. We say that V has *finite gluing* if for every stable and pointed coordinatized curve $(\mathscr{C}, P_\bullet, t_\bullet)$ with a node Q, and modules $W^\bullet$, the space of coinvariants $[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}$ is coherent over $\mathrm{Spec}(\mathbb{C}[\![t]\!])$.

Remark 5.2.4. We make two observations:

(i) We note that if V is C_2-cofinite, then by [DGK22, Corollary 4.2], V has coherent coinvariants and finite gluing. As we shall see, this is also true for VOAs like the Heisenberg which are generated in degree 1, but are not C_2-cofinite.
(ii) By Propagation 4.3.1, if V has finite gluing it follows that $[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}$ is actually free over $\mathbb{C}[\![q]\!]$.

We begin with an auxiliary result.

Lemma 5.2.5. *Let V be a C_1-cofinite VOA that satisfies smoothing and such that $[W^\bullet[\![q]\!]]_{(\mathscr{C},P_\bullet,t_\bullet)}$ and $[(W^\bullet \otimes \mathfrak{A})[\![q]\!]]_{(\widetilde{\mathscr{C}},P_\bullet \sqcup Q_\pm, t_\bullet \sqcup s_\pm)}$ are coherent over S. Then the map $[\alpha]$ defined in Corollary 5.2.1 is an isomorphism.*

Proof. Since V is C_1-cofinite, Lemma 4.4.4 ensures that $[\alpha_0]$ is an isomorphism. Since the source and target of $[\alpha]$ is finitely generated and the target is locally free (see Remark 5.2.4 (ii)), Nakayama's lemma ensures that $[\alpha]$ is an isomorphism as well. □

To state the next results, we shall refer to the moduli stacks $\widehat{\overline{\mathcal{M}}}_{g,n}$, parametrizing families of stable pointed curves of genus g with coordinates, and $\overline{\mathcal{J}}_{g,n}$, of stable pointed curves of genus g with first order tangent data, and projection maps $\widehat{\overline{\mathcal{M}}}_{g,n} \to \overline{\mathcal{J}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ discussed in detail in [DGT24, §2]. Recall the notation from Remark 4.1.2.

Corollary 5.2.6. *Let $W^1, \ldots, W^n$ be simple modules over a C_1-cofinite vertex operator algebra V, such that coinvariants are coherent for curves of genus g, and such that $\mathfrak{A}_d(V)$ admit strong identity elements for all $d \in \mathbb{Z}_{\geq 0}$. Then sheaves of coinvariants are locally free, giving rise to a vector bundle $\mathbb{V}_g(V; W^\bullet)^{\overline{\mathcal{J}}_{g,n}}$ on $\overline{\mathcal{J}}_{g,n}$. If the conformal dimensions of $W^1, \ldots, W^n$ are rational, these sheaves define vector bundles $\mathbb{V}_g(V; W^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$.*

Proof. Since a sheaf of $\mathcal{O}_S$-modules is locally free if and only if it is coherent and flat, in order to show a coherent sheaf $[W^\bullet]_{\mathcal{C}}$ is locally free, it suffices to show that it is flat. For this, we can use the valuative criteria of [Gro67, Thm 11.8.1, §3] to reduce to the case that our base scheme is $S = \mathrm{Spec}(\mathbb{C}[\![t]\!])$. By [Har77, Ex. II.5.8], since S is Noetherian and reduced, and since formation of coinvariants commutes with base change, by Lemma 4.1.1, it suffices to check that vector spaces of coinvariants have the same dimension over all pointed and coordinatized curves.

Our strategy for checking this condition holds is to argue by induction on the number of nodes, reducing to the base case where the curve has no nodes.

To take the inductive step, following the notation of Corollary 5.2.1, let $\mathscr{C}_0 \to \mathrm{Spec}(k)$ be a nodal curve with $k+1$ nodes, and let $\mathscr{C} \to \mathrm{Spec}(\mathbb{C}[\![q]\!])$ be a smoothing family with $\mathscr{C}_0$ the special fiber. By Proposition 4.3.4 and by Lemma 5.2.5, we deduce that $[\alpha]$ is an isomorphism, so that the dimension of the space of coinvariants associated with $\mathscr{C}_0$ agrees with the dimension of the space of coinvariants for the partial normalization $\widetilde{\mathscr{C}}_0$, a curve with k nodes. Therefore, by induction, the vector space $[W^\bullet]_{(\widetilde{\mathscr{C}}_0, P_\bullet, t_\bullet)}$ has the same dimension as the vector space of coinvariants associated with a smooth curve.

We are then left to show that spaces of coinvariants associated with smooth curves of the same genus have the same dimensions. This holds since coinvariants $[W^\bullet]_{\mathcal{C}}$ are by assumption coherent, and moreover, when restricted to families of smooth coordinatized curves, they define a sheaf which admits a projectively flat connection [FBZ04, DGT21]. We have shown that $[W^\bullet]_{\mathcal{C}}$ is flat, giving rise to a coherent and locally free sheaf on $\widehat{\overline{\mathcal{M}}}_{g,n}$. As shown in [DGT24], this sheaf of coinvariants descends to a sheaf of coinvariants $\mathbb{V}_g(V; W^\bullet)^{\overline{\mathcal{J}}_{g,n}}$ on $\overline{\mathcal{J}}_{g,n}$. Moreover, for any collection of simple V-modules $W^\bullet$ with rational conformal weights, as is explained in [DGT24, §8.7.1], the sheaves are independent of coordinates and will further descend to vector bundles on $\overline{\mathcal{M}}_{g,n}$, denoted $\mathbb{V}_g(V; W^\bullet)$. □

Remark 5.2.7. We note the following consequences of Corollary 5.2.6:

(a) For a collection of simple modules over a C_2-cofinite VOA, the sheaf of coinvariants will give vector bundles $\mathbb{V}_g(V; W^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ whenever the algebras $\mathfrak{A}_d(V)$ admit strong identity elements. To see this, we note that the coinvariants will be coherent by [DGK22], and by [Miy04, Corollary 5.10] any simple module over a C_2-cofinite V has rational conformal weight.

(b) Combining Remark 3.4.6 with Corollary 5.2.6 one may show that sheaves of coinvariants from C_2-cofinite and rational VOAs define vector bundles on $\overline{\mathcal{M}}_{g,n}$, recovering [DGT24, VB Corollary].

(c) By [DG23], sheaves defined by simple modules over VOAs that are generated in degree 1 are coherent over rational curves. If V satisfies smoothing, such sheaves of coinvariants descend to vector bundles $\mathbb{V}_0(V; W^\bullet)^{\overline{\mathcal{J}}_{0,n}}$ on $\overline{\mathcal{J}}_{0,n}$. If the conformal dimensions of the modules are in $\mathbb{Q}$, they further descend to vector bundles $\mathbb{V}_0(V; W^\bullet)$ on $\overline{\mathcal{M}}_{0,n}$. Moreover, by [DG23], these bundles are globally generated. We refer to Sect. 7 and Corollary 7.4.1 for an application of this using the Heisenberg VOA.

6. Higher Zhu Algebras and Mode Transition Algebras

Recall that if any of the equivalent properties of Definition/Lemma 3.3.1 hold, we say that $\mathscr{I}_d \in \mathfrak{A}_d$ is a strong identity element. Here we prove Theorem 6.0.1, one of our two main results. In order to formulate it, we introduce the map

$$\mu_d : \mathfrak{A}_d \to \mathsf{A}_d, \qquad \mu_d(\alpha \otimes u \otimes \beta) = [\alpha u \beta]_d.$$

This map is well-defined and fits into an exact sequence (see Lemma B.3.1)

$$\mathfrak{A}_d \xrightarrow{\mu_d} \mathsf{A}_d \xrightarrow{\pi_d} \mathsf{A}_{d-1} \longrightarrow 0. \tag{20}$$

Theorem 6.0.1. *(a) If the mode transition algebra $\mathfrak{A}_d$ admits an identity element element, then the map μ_d in (20) is injective, and the sequence splits, yeilding an isomorphism $\mathsf{A}_d \cong \mathfrak{A}_d \times \mathsf{A}_{d-1}$ as rings. In particular, if $\mathfrak{A}_j$ admits an identity element for every $j \leq d$, then $\mathsf{A}_d \cong \mathfrak{A}_d \oplus \mathfrak{A}_{d-1} \oplus \cdots \oplus \mathfrak{A}_0$.*

(b) If $\mathfrak{A}_d$ admits a strong identity element for all $d \in \mathbb{N}$, so that smoothing holds for V, then given any generalized Verma module $W = \Phi^{\mathsf{L}}(W_0) = \bigoplus_{d\in\mathbb{N}} W_d$ where L_0 acts on W_0 as a scalar with eigenvalue $c_W \in \mathbb{C}$, there is no proper submodule $Z \subset W$ with $c_Z - c_W > 0$ for every eigenvalue c_Z of L_0 on Z (see Remark 6.0.2).

We note that Theorem B.3.3 specializes to Part ((a)) of Theorem 6.0.1. It therefore remains to prove Part ((b)) of Theorem 6.0.1.

Proof. We say that an induced admissible module $W = \Phi^{\mathsf{L}}(W_0)$ has the LCW property if L_0 acts on W_0 as a scalar with eigenvalue $c_W \in \mathbb{C}$, and there is no proper submodule $Z \subset W$ with $c_Z - c_W > 0$ for every eigenvalue c_Z of L_0 on Z. Suppose for contradiction that V admits a module $W = \Phi^{\mathsf{L}}(W_0)$, and W does not have the LCW property. We will show that there must be a $d \in \mathbb{N}$ such that $\mathfrak{A}_d$ is not unital, contradicting our assumptions.

By hypothesis, W has a proper submodule Z with $c_Z - c_W > 0$ for every eigenvalue c_Z of L_0 on Z. In particular, Z is not induced in degree zero over A. Let z_d be any homogeneous element in Z of smallest degree $d > 0$, so that $z_d \in W_d$. By assumption $\mathfrak{A}_d$ is unital, with unity $\mathfrak{u}_d = \sum_i \alpha_i \otimes 1 \otimes \beta_i$, where each α_i has degree d and each β_i has degree $-d$. The action of $\mathfrak{A}$ on W restricts to an action of $\mathfrak{A}_d$ on W_d, and since $\mathfrak{u}_d$ is the unity of $\mathfrak{A}_d$ we have

$$\mathfrak{A}_d \times W_d \longrightarrow W_d, \qquad (\mathfrak{u}_d, z_d) \mapsto \mathfrak{u}_d \star z_d = z_d.$$

Unraveling the definition of $\star$ and its associativity properties we have $\mathfrak{u}_d \star z_d = \sum_i \alpha_i \cdot (\beta_i \cdot z_d)$. But now since the degree of $\beta_i \cdot z_d$ is zero and Z is a submodule, we have that $\beta_i \cdot z_d \in Z \cap W_0 = 0$, since Z does not have a degree zero component. It then follows that $z_d = \mathfrak{u}_d \star z_d = 0$, giving a contradiction since we assumed $z_d \neq 0$. □

Remark 6.0.2. Although the eigenvalues c_Z and c_W are in general complex numbers, the difference $c_Z - c_W$ is always an integer, hence it makes sense to require that this number be positive. In fact, every eigenvalue of the action of L_0 on W will be obtained by shifting c_W by a non-negative integer. The condition $c_Z - c_W > 0$ coincides then with $c_Z \neq c_W$. We remark that when V is C_2-cofinite, then the eigenvalues of L_0 are necessarily rational numbers [Miy04].

7. Mode Transition Algebra for the Heisenberg VOA

In this section we describe the mode transition algebras for the Heisenberg VOA. This result is stated in Proposition 7.2.1 and, as a consequence, in Sect. 7.3 we obtain that [AB23a, Conjecture 8.1] holds. We refer [FBZ04, LL04, Mil08, BVWY19a, AB23a] for more details about the Heisenberg VOA, denoted π, $V_{\hat{\mathfrak{h}}}(1,\alpha)$, $M_a(1)$ and $M(1)_a$ in the literature, and which we next briefly describe. We will use the notation π as in [FBZ04, Section 4.3].

7.1. Background on the Heisenberg VOA. Let $\mathfrak{h} = H\mathbb{C}(\!(t)\!) \oplus k\mathbb{C}$ be the extended Heisenberg algebra and consider the Heisenberg VOA $V = \pi$. Let $U_1(\mathfrak{h})$ denote the quotient of the universal enveloping algebra $U(\mathfrak{h})$ by the two sided ideal generated by $k-1$. Following [FBZ04, Section 4.3] the Lie algebra $\mathfrak{L}(V)^{\mathsf{L}}$ is naturally embedded inside

$$\overline{U(\mathfrak{h})}^{\mathsf{L}} := \varprojlim \frac{U_1(\mathfrak{h})}{U_1(\mathfrak{h}) \circ Ht^N\mathbb{C}[t]}.$$

The map is induced by $(b_{-1})_{[n]} \mapsto Ht^n$. This embedding induces a natural isomorphism between $\mathscr{U}^{\mathsf{L}}$ and $\overline{U(\mathfrak{h})}^{\mathsf{L}}$ which translates the filtration on $\mathscr{U}^{\mathsf{L}}$ into the canonical filtration on $\overline{U(\mathfrak{h})}^{\mathsf{L}}$ induced by the filtration on $\mathbb{C}(\!(t)\!)$ given by $F^p\mathbb{C}(\!(t)\!) = t^{-p}\mathbb{C}[t^{-1}]$.

A similar construction holds for $\mathfrak{L}(V)^{\mathsf{R}}$ and $\mathscr{U}^{\mathsf{R}}$, where the extended Heisenberg algebra $\mathfrak{h} = H\mathbb{C}(\!(t)\!) \oplus \mathbb{C}$ is replaced by $\mathfrak{h} = H\mathbb{C}(\!(t^{-1})\!) \oplus \mathbb{C}$.

The sub ring $\mathscr{U}$ of $\mathscr{U}^{\mathsf{L}}$ and $\mathscr{U}^{\mathsf{R}}$ has a natural gradation induced by $\deg(Ht^n) = -n$. We can then deduce that the associated zero mode algebra $\mathfrak{A}_0$ is isomorphic to the commutative ring $\mathbb{C}[x]$, where the element $(b_{-1})_{[0]} = H \in \mathscr{U}_0$ is identified with the variable x. Combining these results we can explicitly compute all the mode transition algebras.

7.2. Mode Transition Algebras for the Heisenberg VOA. We can now state and prove the main result of this section.

Proposition 7.2.1. *There is a natural identification* $\mathfrak{A}_d(\pi) \cong \mathrm{Mat}_{p(d)}(\mathbb{C}[x])$, *where* $p(d)$ *is the number of ways to decompose* d *into a sum of positive integers. In particular* $\mathfrak{A}_d$ *is unital for every* $d \in \mathbb{N}$.

Proof. Denote by $P(d)$ the set of partitions of d into positive integers, so that $|P(d)| = p(d)$. We represent every element $[r_1|\cdots|r_n] = \boldsymbol{r} \in P(d)$ by a decreasing sequence of positive integers $r_1 \geq \cdots \geq r_n \geq 1$ such that $\sum_i r_i = d$ and for some $n \in \mathbb{N}$. For every pair $(\boldsymbol{r},\boldsymbol{s}) \in P(d)^2$, we denote by $\varepsilon_{\boldsymbol{r},\boldsymbol{s}}$ the element in $\mathfrak{A}_d$ given by

$$Ht^{-r_1} \circ \cdots \circ Ht^{-r_n} \otimes 1 \otimes Ht^{s_m} \circ \cdots \circ Ht^{s_1}.$$

From the explicit description of $\mathscr{U}$ given above, and the fact that the Zhu algebra $\mathsf{A} = \mathbb{C}[x]$ at level zero is Abelian, we have that the set whose elements are $\varepsilon_{\boldsymbol{r},\boldsymbol{s}}$ freely generates $\mathfrak{A}_d$ as an A-module. Moreover, using a computation similar to Example 3.2.3, one may show that

$$Ht^{s_m} \circ \cdots \circ Ht^{s_1} \circledast Ht^{-r_1} \circ \cdots \circ Ht^{-r_m} = \begin{cases} ||\mathbf{r}|| & \text{if } \mathbf{s} = \mathbf{r} \\ 0 & \text{otherwise,} \end{cases}$$

where $||\boldsymbol{r}||$ is a non-zero, positive integer entirely depending on $\boldsymbol{r}$. It then follows that

$$\varepsilon_{\boldsymbol{r}',\boldsymbol{s}} \star \varepsilon_{\boldsymbol{r},\boldsymbol{s}'} = \begin{cases} ||\boldsymbol{r}||\varepsilon_{\boldsymbol{r}',\boldsymbol{s}'} & \text{if } \boldsymbol{s} = \boldsymbol{r} \\ 0 & \text{otherwise.} \end{cases}$$

By identifying $\varepsilon_{\boldsymbol{r},\boldsymbol{s}}$ with the element of $\mathrm{Mat}_{p(d)}(\mathbb{C})$ having $\sqrt{||\boldsymbol{r}||}\sqrt{||\boldsymbol{s}||}$ in the $(\boldsymbol{r},\boldsymbol{s})$-entry, and zero otherwise, the above description gives an isomorphism of rings between $\mathfrak{A}_d$ and $\mathbb{C}[x] \otimes \mathrm{Mat}_{p(d)}(\mathbb{C}) = \mathrm{Mat}_{p(d)}(\mathbb{C}[x])$, as is claimed. □

Example 7.2.2. We can explicitly compute the coefficient $||\boldsymbol{r}||$ appearing in the proof of Proposition 7.2.1. Let $\boldsymbol{r} = [r_1|\ldots|r_n]$ be a partition of d consisting of s many distinct elements $r_{i_1}, \ldots, r_{i_s}$ (at most $s = d$). For every $j \in \{1, \ldots, s\}$, let m_j be the multiplicity of r_{i_j} in $\boldsymbol{r}$. Then we have

$$||\boldsymbol{r}|| = \prod_{i=1}^{n} r_i \cdot \prod_{j=1}^{s} m_j!.$$

For instance $||[1|\cdots|1]|| = d!$ and $||[d]|| = d$. Moreover $||[r_1|\cdots|r_d]|| = r_1 \cdots r_d$ if the r_i's are all distinct.

7.3. The Conjecture of Barron and Addabbo. We now prove [AB23a, Conj. 8.1].

Corollary 7.3.1. *For all $d \in \mathbb{N}$, one has that $\mathsf{A}_d(\pi) \cong \mathrm{Mat}_{p(d)}(\mathbb{C}[x]) \oplus \mathsf{A}_{d-1}(\pi)$.*

Proof. This follows from Proposition 7.2.1 and Part *(a)* of Theorem 6.0.1. □

Remark 7.3.2. By [BVWY19a, Remark 4.2], $\mathsf{A}_0(\pi) \cong \mathbb{C}[x]$, $\mathsf{A}_1(\pi) \cong \mathbb{C}[x] \oplus \mathsf{A}_0(\pi)$, and by [AB23a, Theorem 7.1], $\mathsf{A}_2(\pi) \cong \mathrm{Mat}_{p(2)}(\mathbb{C}[x]) \oplus \mathsf{A}_1(\pi)$.

7.4. Vector Bundles from the Heisenberg VOAs. We now equip π with a conformal vector ω, so that it becomes a VOA. The following result shows that the application of Theorem 6.0.1 produces new examples, beyond the well-studied case of sheaves of coinvariants defined by rational and C_2-cofinite VOAs.

Let $\overline{\mathcal{J}}_{0,n}$ be the stack parametrizing families of stable pointed curves of genus zero with first order tangent data, and recall that the forgetful map $\pi : \overline{\mathcal{J}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ makes $\overline{\mathcal{J}}_{0,n}$ a $\mathbb{G}_m^{\oplus n}$-torsor over $\overline{\mathcal{M}}_{0,n}$.

Corollary 7.4.1. *Sheaves of coinvariants defined by simple modules over the Heisenberg VOA form globally generated vector bundles on $\overline{\mathcal{J}}_{0,n}$. If conformal dimensions of modules are in $\mathbb{Q}$, these descend to form globally generated vector bundles on $\overline{\mathcal{M}}_{0,n}$.*

Proof. By Proposition 7.2.1, the mode transition algebras for the Heisenberg VOAs are unital. Moreover, the formula of the star product implies that these are strong identity elements. Hence by Theorem 6.0.1, the Heisenberg VOA satisfies smoothing. Since the Heisenberg VOA is by definition generated in degree 1, the assertion follows from Corollary 5.2.6, as described in Remark 5.2.7 (c). □

Remark 7.4.2. Unlike bundles of coinvariants given by representations of rational and C_2-cofinite VOAs, higher Chern classes of bundles on $\overline{\mathcal{M}}_{g,n}$ from Corollary 5.2.6 (like those on $\overline{\mathcal{M}}_{0,n}$ from Corollary 7.4.1) are elements of the tautological ring since we do not know if they satisfy factorization, and hence we do not know that the Chern characters form a semisimple cohomological field theory as in [MOP+17, DGT22].

8. Mode Transition Algebras for Virasoro VOAs

For $c \in \mathbb{C}$, by $\mathrm{Vir}_c = M_{c,0}/< L_{-1}1 >$ we mean the (not necessarily simple) Virasoro VOA of central charge $c \in \mathbb{C}$. We refer to [Wan93] for further details about this VOA, which is denoted M_c by the author.

8.1. Vir_c. By [Wan93], when $c \neq c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$, then Vir_c is a simple VOA, but it is not rational or C_2-cofinite. When $c = c_{p,q}$, the VOA Vir_c is not simple, but its simple quotient L_c will be rational and C_2-cofinite, and therefore satisfy smoothing. We therefore only consider Vir_c, for any values of c, and not L_c.

Proposition 8.1.1. *Let* Vir_c *be the Virasoro VOA.*

(a) The first mode transition algebra $\mathfrak{A}_1(\mathrm{Vir}_c)$ *is not unital, and so* Vir_c *does not satisfy smoothing.*
(b) The kernel of the canonical projection $\mathsf{A}_1(\mathrm{Vir}_c) \to \mathsf{A}_0(\mathrm{Vir}_c)$ *is isomorphic to* $\mathfrak{A}_1(\mathrm{Vir}_c)$.

Proof. We first prove *(a)*. By [Wan93, Lemma 4.1], one has $\mathsf{A}_0(\mathrm{Vir}_c) \cong \mathbb{C}[t]$, where the class of $(L_{-2}1)_{[1]}$ is mapped to the generator t.

Here, as in the Heisenberg case, $\mathfrak{L}(V)^{\mathsf{f}}_{\pm 1}$ is a one dimensional vector space, with generators denoted $u_{\pm 1}$, so that $\mathfrak{A}_1(\mathrm{Vir}_c) = u_1 \mathsf{A}_0(\mathrm{Vir}_c) u_{-1}$. We can choose $u_1 = (L_{-2}1)_{[0]}$ and $u_{-1} = (L_{-2}1)_{[2]}$, and to understand the multiplicative structure of $\mathfrak{A}_1(\mathrm{Vir}_c)$ we are only left to compute $[u_{-1}, u_1]$. Since $L_{-2}1$ is the conformal vector of Vir_c, we can identify $(L_{-2}1)_{[n]}$ with the element $\mathcal{L}_{n-1}$ of the Virasoro algebra, and the bracket of $L(\mathrm{Vir}_c)$ coincides with the bracket in the Virasoro algebra. Hence we obtain

$$[u_{-1}, u_1] = [(L_{-2}1)_{[2]}, (L_{-2}1)_{[0]}] = [\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0 = 2(L_{-2}1)_{[1]}.$$

We then have an identification of $\mathfrak{A}_1(\mathrm{Vir}_c)$ with $(\mathbb{C}[t], +, \star)$, where $+$ denotes the usual sum of polynomials, while $f(t) \star g(t) = 2tf(t)g(t)$. In particular, this implies that $\mathfrak{A}_1(\mathrm{Vir}_c)$ is not unital.

We now show *(b)*. By [Wan93], $\mathsf{A}_0(\mathrm{Vir}_c)$ is generated by $L_{-2}\mathbf{1} + O_0(V)$ and $L_{-2}^2\mathbf{1} + O_0(V)$ so that

$$\mathsf{A}_0(\mathrm{Vir}_c) \cong \mathbb{C}[x, y]/(y - x^2 - 2x) \cong \mathbb{C}[x],$$

$$L_{-2}\mathbf{1} + O_0(V) \mapsto x + (q_0(x, y)), \qquad L_{-2}^2\mathbf{1} + O_0(V) \mapsto y + (q_0(x, y)),$$

where $q_0(x, y) = y - x^2 - 2x$. By [BVWY20, Theorem 4.7], $\mathsf{A}_1(\mathrm{Vir}_c)$ is generated by $L_{-2}\mathbf{1} + O_1(V)$ and $L_{-2}^2\mathbf{1} + O_1(V)$, and by [BVWY20, Theorem 4.11] on has that

$$\mathsf{A}_1(\mathrm{Vir}_c) \cong \mathbb{C}[x, y]/((y - x^2 - 2x)(y - x^2 - 6x + 4)),$$

$$L_{-2}\mathbf{1} + O_1(V) \mapsto x + (q_0(x, y)q_1(x, y)), \qquad L_{-2}^2\mathbf{1} + O_1(V) \mapsto y + (q_0(x, y)q_1(x, y)),$$

where $q_0(x, y) = y - x^2 - 2x$ and $q_1(x, y) = y - x^2 - 6x + 4$ (see also [BVWY20, §5]). With the change of variables $X = y - x^2 - 6x + 4$ and $Y = y - x^2 - 2x$, one has

$$\mathsf{A}_1(\mathrm{Vir}_c) = \frac{\mathbb{C}[X, Y]}{XY} \qquad \text{and} \qquad \mathsf{A}_0(\mathrm{Vir}_c) = \mathbb{C}[X],$$

so that the kernel of the projection $\mathsf{A}_1(\mathrm{Vir}_c) \to \mathsf{A}_0(\mathrm{Vir}_c)$ is identified with the ideal K_1 generated by Y inside $\mathsf{A}_1(\mathrm{Vir}_c)$. Since $XY = 0$, the ideal K_1 is isomorphic to

$(Y\mathbb{C}[Y], +, \cdot)$. Furthermore, this algebra is isomorphic to the algebra $(\mathbb{C}[t], +, \star)$ through the assignment $Yf(Y) \mapsto f(2t)$. This shows that, abstractly, $\mathfrak{A}_1(\mathrm{Vir}_c)$ is identified with the kernel of $\mathsf{A}_1(\mathrm{Vir}_c) \to \mathsf{A}_0(\mathrm{Vir}_c)$.

We now see directly that this identification is provided by the natural map $\mu_1 : \mathfrak{A}_1 \to \mathsf{A}_1(V)$, which is induced by $(L_{-2}1)_{[0]} \otimes 1 \otimes (L_{-2}1)_{[2]} \mapsto [(L_{-2}1)_{[0]}(L_{-2}1)_{[2]}]$ as in Lemma B.3.1. To check that indeed $\mathfrak{A}_1(\mathrm{Vir}_c)$ naturally identifies with the kernel of $\mathsf{A}_1(\mathrm{Vir}_c) \to \mathsf{A}_0(\mathrm{Vir}_c)$, it is enough to show that

$$\tilde{Y} - 2(L_{-2}1)_{[0]}(L_{-2}1)_{[2]} \in N^2\mathscr{U}_0,$$

where $\tilde{Y}$ is any lift of Y to $\mathscr{U}_0$. We choose

$$\tilde{Y} = (L_{-2}L_{-2}1)_{[3]} - (L_{-2}1)_{[1]}(L_{-2}1)_{[1]} - 2(L_{-2}1)_{[1]}.$$

To simplify the notation, we will now write $\mathcal{L}_n$ to denote $(L_{-2}1)_{[n+1]}$. Using the Virasoro relations we obtain that this is the same as

$$\begin{aligned}
\tilde{Y} &= 2\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n + \mathcal{L}_{-1}\mathcal{L}_1 + \mathcal{L}_1\mathcal{L}_{-1} + \mathcal{L}_0\mathcal{L}_0 - \mathcal{L}_0\mathcal{L}_0 - 2\mathcal{L}_0 \\
&= 2\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n + \mathcal{L}_{-1}\mathcal{L}_1 + \mathcal{L}_1\mathcal{L}_{-1} - 2\mathcal{L}_0 \\
&= 2\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n + 2\mathcal{L}_{-1}\mathcal{L}_1 + 2\mathcal{L}_0 - 2\mathcal{L}_0 \\
&= 2\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n + 2\mathcal{L}_{-1}\mathcal{L}_1 \\
&= 2\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n + 2(L_{-2}1)_{[0]}(L_{-2}1)_{[2]},
\end{aligned}$$

and since $\sum_{n\geq 2} \mathcal{L}_{-n}\mathcal{L}_n \in N^2\mathscr{U}_0$, the proof is complete. □

9. Questions

Here we ask a few other questions that arise from this work.

9.1. Not Rational and Strongly Generated in Higher Degree. Keeping in mind the example of the Virasoro VOA from Sect. 8 and Theorem 6.0.1, we ask the following:

Question 9.1.1. For V a C_2-cofinite and non-rational VOA, not generated in degree 1, can one always find a pair $Z \subset W$ where $W = \Phi^{\mathsf{L}}(W_0)$ is induced by an indecomposable $\mathsf{A}_0(V)$-module W_0, such that L_0 acts on W_0 as a scalar with eigenvalue $c_W \in \mathbb{C}$, and a proper submodule $Z \subset W$, with $c_Z - c_W > 0$ for every eigenvalue c_Z of L_0 on Z.

In Sect. 9.1.2 we provide an example of such a pair of modules $Z \subset W$ for the triplet vertex operator algebra $\mathcal{W}(p)$. This particular example was suggested to us in a communication with Thomas Creutzig. Simon Wood gave us a proof of Claim 9.1.3, a crucial detail for this example. The features of such an example (and that it should exist for the triplet) were described to us by Dražen Adamović.

9.1.2. Triplet VOAs For $p \geq 2$, let $\mathcal{W}(p)$ denote the triplet vertex operator algebra. There are $2p$ simple $\mathcal{W}(p)$-modules X_s^+, and X_s^-, for $1 \leq s \leq p$. Following [TW13, Eq (2.39)], we write $\overline{X}_s^{\pm}$ for the quotient $\mathsf{A}_0(X_s^{\pm}) = X_s^{\pm}/I_0(X_s^{\pm})$ which are simple modules over Zhu algebra $\mathsf{A} = \mathsf{A}_0(\mathcal{W}(p))$. And in this case, one also has in the notation of [BVWY19a], that $\Omega_0(X_s^{\pm}) = (X_s^{\pm})_0 = \overline{X}_s^{\pm}$. In particular, since $\overline{X}_s^{\pm}$ is an A-module, we may consider $\Phi^{\mathsf{L}}(X_s^{\pm})$. Moreover, using for instance [TW13, Eq (3.8)], the eigenvalues of the action of L_0 on the indecomposable modules $\overline{X}_s^{\pm}$, i.e. the conformal weights, satisfy $cw(X_{p-s}^-) > cw(X_s^+)$. The induced module $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ can be identified with a quotient of the projective cover of X_s^+, as follows. By [NT11, Proposition 4.5] (see also [TW13]) the projective cover P_s^+ of X_s^+ has socle filtration of length three consisting of submodules $S_0 \subset S_1 \subset S_2 = P_s^+$ with $S_0 \cong X_s^+ \cong S_2/S_1$ and $S_1/S_0 \cong 2X_{p-s}^-$.

Claim 9.1.3. $\Phi^{\mathsf{L}}(\overline{X}_s^+) \cong P_s^+/X_s^+$

Proof. The A-module $\overline{X}_s^+$ is indecomposable, and as Φ^{L} takes indecomposable modules to indecomposable modules (eg. [DGK22]), one has that $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ is an indecomposable admissible $\mathcal{W}(p)$ module. It follows that $\overline{X}_s^+$ will be the weight space of least conformal weight in $\Phi^{\mathsf{L}}(\overline{X}_s^+)$, and as X_s^+ is generated by its lowest weight space $\overline{X}_s^+$, we get a canonical surjective map $\Phi^{\mathsf{L}}(\overline{X}_s^+) \to X_s^+$. By projectivity, the map from the projective cover $P_s^+ \to X_s^+$ lifts to a map $P_s^+ \to \Phi^{\mathsf{L}}(\overline{X}_s^+)$. As this map is surjective on the least weight space, the weight of X_s^+, and $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ is generated by this subspace by construction, the map $P_s^+ \to \Phi^{\mathsf{L}}(\overline{X}_s^+)$ is surjective and so $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ is a quotient of P_s^+.

The kernel of this quotient must contain the socle (which isomorphic to X_s^+), otherwise $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ would have two composition factors isomorphic to X_s^+ contradicting the size of its lowest weight space. The kernel cannot be larger, otherwise $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ would admit a nontrivial extension by X_{p-s}^- (which as noted, has greater conformal weight than X_s^+). Note that if we had such an extension $0 \to X_{p-s}^- \to E \to \Phi^{\mathsf{L}}(\overline{X}_s^+) \to 0$, the lowest weight spaces of E and $\Phi^{\mathsf{L}}(\overline{S}_s^+)$ would be isomorphic, producing a universal map $\Phi^{\mathsf{L}}(\overline{X}_s^+) \to E$ whose composition with the map in the above sequence would be the universal map from $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ to itself – that is, the identity. Consequently we would have a splitting of our exact sequence and the extension would be trivial. Thus $\Phi^{\mathsf{L}}(\overline{X}_s^+)$ is isomorphic to P_s^+/X_s^+. □

In particular, by Claim 9.1.3, $W = \Phi^{\mathsf{L}}(\overline{X}_s^+) = S_2/S_0$ would have a sub-module isomorphic to $Z = S_1/S_0 \cong 2X_{p-s}^-$, and the conformal weight $cw(Z) = cw(X_{p-s}^-)$ would then be strictly larger than the conformal weight $cw(W) = cw(\overline{X}_s^+)$.

Proposition 9.1.4. $\mathcal{W}(p)$ *does not satisfy smoothing.*

Proof. The example shows by Theorem 5.0.3 and Theorem 6.0.1 that the triplet does not satisfy smoothing. □

9.1.5. More General Triplet In [FGST06, AM07, AM10, AM11] the more general triplet VOAs $\mathcal{W}_{p_+,p_-}$ with $p_\pm \geq 2$ and $(p_+, p_-) = 1$ are studied. From their results, for $p_+ = 2$ and p_- odd, the $\mathcal{W}_{p_+,p_-}$ are C_2-cofinite and not rational. We would like to know the answer to Question 9.1.1 for this family of VOAs.

9.1.6. Other C_2-Cofinite, Non-Rational VOAs from Extensions In [CKL20] the authors discover three new series of C_2-cofinite and non-rational VOAs, via application of the vertex tensor category theory of [HLZ06,HLZ14], which are not directly related to the triplets. They also list certain modules for these examples. We would like to know the answer to Question 9.1.1 for these new families of VOAs.

9.2. Local Freeness in Case V does Not Satisfy Smoothing.

Question 9.2.1. Are there particular choices of modules $W^\bullet$ over a V that do not satisfy smoothing, for which sheaves $\mathbb{V}(V; W^\bullet)$ form vector bundles on $\overline{\mathcal{M}}_{g,n}$?

By Corollary 5.2.6, if V satisfies smoothing, and if the sheaves of coinvariants are coherent, they form vector bundles. However, if V does not satisfy smoothing, it is still an open question about whether these sheaves are locally free. For instance one could ask this for the triplet VOAs, which do not satisfy smoothing, but are C_2-cofinite, so their representations define coherent sheaves on $\overline{\mathcal{M}}_{g,n}$.

9.3. Relation Between $\mathfrak{A}(V)$ and $A^\infty(V)$. As discussed in Sect. 6, the d-th mode transition algebra $\mathfrak{A}_d$ is related to the higher level Zhu algebra A_d via the map

$$\mu_d\colon \mathfrak{A}_d \to \mathsf{A}_d, \qquad \mu_d(\alpha \otimes u \otimes \beta) = [\alpha u \beta]_d. \tag{21}$$

In [Hua20,Hua21,Hua23], an associative algebra $A^\infty(V)$, together with a series of subalgebras $A^d(V)$, is defined and studied. As every $A^d(V)$ contains as subalgebras the higher level Zhu algebras $\mathsf{A}_0, \ldots, \mathsf{A}_d$ [Hua20, Remark 4.5], it is natural then to ask:

Question 9.3.1. Does (21) extend to a map $\bigoplus_{0\le i,j\le d} \mathfrak{A}_{i,-j} \to A^d(V)$, compatible with the action of these two algebras on $\bigoplus_{i=0}^d W_i$ for an $\mathbb{N}$-graded admissible V-module W?

If this were true, we suspect that there may be a right exact sequence

$$\bigoplus_{0\le i,j\le d} \mathfrak{A}_{i,-j} \xrightarrow{-\mu_d} A^d(V) \xrightarrow{\pi_d} A^{d-1}(V) \longrightarrow 0.$$

In this case, one would expect a map to $A^\infty(V)$ from a suitable completion of $\mathfrak{A}(V)$.

9.4. The Category of $\mathfrak{A}(V)$-Modules. In Sect. 3.2 the action of $\mathfrak{A}(V)$ on any $\mathbb{N}$-graded admissible V-module is defined. This leads one to ask the following:

Question 9.4.1. Does the above action define an equivalence of categories between $\mathbb{N}$-graded admissible V-modules and $\mathbb{N}$-graded $\mathfrak{A}(V)$-modules?

As simple modules and their morphisms are determined by their degree 0 parts thought of as $\mathsf{A}(V)$-modules, it seems reasonable to expect that such an equivalence of categories might arise in the case of rational V. On the other hand, we feel that this seems unlikely to hold for general V. We refer to [DGK24] for a description of Morita equivalences between $\mathfrak{A}_d(V)$-modules and $\mathsf{A}_d(V)$-modules.

9.5. Generalized Constructions. In Sect. A.9 the notion of triples of associative algebras is introduced, and to a good triple (see Definition A.9.1) we associate many of the standard notions affiliated with a VOA from higher level Zhu algebras to mode transition algebras (see Sect. B.1 and Sect. B.2). Some of the results proved here apply in this more general context. For instance, as was already noted in the introduction, the exact sequence in (1), and Part (a) of Theorem 6.0.1 hold in this generality. It would be interesting to further develop this theory, and it is therefore natural to ask the following question:

Question 9.5.1. What are other examples of generalized (higher level) Zhu algebras and generalized mode transition algebras, beyond the context of VOAs?

Acknowledgements. We would like to thank Dražen Adamović, Katrina Barron, Thomas Creutzig, Haisheng Li, Jianqi Liu, Antun Milas, Rahul Pandharipande, Dennis Sullivan, and Simon Wood for helpful discussions and for answering our questions. Thomas Creutzig and Simon Wood gave crucial assistance with the details of Sect. 9.1.2. We thank Katrina Barron for her valuable feedback at various points. We would also like to thank the anonymous referees for their comments, questions, and corrections. Most of all, we thank Yi-Zhi Huang who encouraged us from the beginning to consider questions regarding sewing, when rationality is not assumed. Damiolini was supported by NSF DMS – 2401420, Gibney was supported by NSF DMS – 2200862, and Krashen was supported by NSF DMS – 1902237.

Appendix A. Split Filtrations

This appendix contains a number of details about graded and filtered completions, and their relationships to one another. These serve to provide simple definitions of the building blocks of our constructions and uniform proofs of their properties.

A.1. Filtrations. The purpose of this first section is to provide a framework in which we can simultaneously discuss and compare filtered and graded versions of certain constructions. In particular, this will give us a language appropriate for dealing simultaneously with both graded and filtered versions of the universal enveloping algebra of a vertex operator algebra which we recall in Definition 2.4.3.

Definition A.1.1 (Left and right filtrations). Let X be an Abelian group. A left filtration on X is a sequence of subgroups $X_{\leq n} \subset X_{\leq n+1} \subset X$ for $n \in \mathbb{Z}$. Similarly, a right filtration on X is a sequence of subgroups $X_{\geq n} \subset X_{\geq n-1} \subset X$ for $n \in \mathbb{Z}$.

Remark A.1.2. If X has a left filtration of subgroups $X_{\leq n}$ we may produce a right filtration by setting $X_{\geq n} = X_{\leq -n}$. Hence the concepts of left and right filtrations are essentially equivalent. We will work in this section exclusively with left filtrations, but will have use for both left and right filtrations eventually. The reader should therefore keep in mind that the results in this section all have their right counterparts. If X is a graded Abelian group, we can naturally regard it as filtered by setting $X_{\leq n} = \bigoplus_{i \leq n} X_i$.

Notation A.1.3. *If X is a filtered Abelian group and $S \subset X$ is a subset, we write $S_{\leq n}$ to mean $S \cap X_{\leq n}$.*

We will now introduce some concepts which we will use throughout.

Definition A.1.4 (Exhaustive filtration). Let X be a (left) filtered Abelian group. We say that the filtration on X is exhaustive if $\bigcup_n X_{\leq n} = X$ and separated if $\bigcap_n X_{\leq n} = 0$.

Definition A.1.5 (Splittings of filtrations). Given a (left) filtered Abelian group X, we define the associated graded group to be $\mathrm{gr}\, X = \bigoplus_n \left(X_{\leq n}/X_{\leq n-1}\right)$. A splitting of X is defined to be a graded subgroup $X' = \bigoplus_n X'_n \subset X$ with $X'_n \subset X_{\leq n}$ such that for each n, the induced map $X'_n \to (\mathrm{gr}\, X)_n$ is an isomorphism.

Definition A.1.6 (Split-filtered Abelian groups). A split-filtered Abelian group is a filtered Abelian group $(X, \leq)$ together with a graded Abelian group $X' = \bigoplus_n X'_n$, and an inclusion $X' \subset X$ which defines a splitting as in Definition A.1.5.

Notation A.1.7. *For X a split-filtered Abelian group, and $x \in X_{\leq n}$, we write $x_n \in X'_n$ and $x_{<n} \in X_{\leq n-1}$ for the unique elements such that $x = x_n + x_{<n}$.*

Example A.1.8 (Concentrated split filtrations). If X is an Abelian group, with no extra structure, we may define a split-filtered structure on it, $X[d]$, which we refer to as "concentrated in degree d," by:

$$X[d]_{\leq p} = \begin{cases} 0 & \text{if } p < d, \\ X & \text{if } p \geq d. \end{cases} \qquad \text{and} \qquad X[d]'_p = \begin{cases} 0 & \text{if } p \neq d, \\ X & \text{if } p = d. \end{cases}$$

If X is an Abelian group, with no extra structure, we may define the trivial split filtration on X to be $X[0]$.

Example A.1.9. If $X = \bigoplus_n X_n$ is a graded Abelian group, we may also consider it as a split Abelian group with respect to the filtration $X_{\leq n} = \bigoplus_{p \leq n} X_p$. In this case the inclusion of X into itself provides the splitting.

Definition A.1.10 (Split-filtered maps). If X and Y are split-filtered Abelian groups and $d \in \mathbb{Z}$, we say that a group homomorphism $f : Y \to X$ is a map of degree d if $f(Y_{\leq p}) \subset X_{\leq p+d}$ and $f(Y'_p) \subset X'_{p+d}$ for all p.

Definition A.1.11 (Split-filtered subgroups). If X and Y are split-filtered Abelian groups with $Y \subset X$, we say Y is a split-filtered subgroup of X if the inclusion is a degree 0 map of split-filtered Abelian groups.

Lemma A.1.12. *Let $f : X \to Y$ be a degree d homomorphism of split-filtered Abelian groups. Then* $\ker f$ *is a split-filtered subgroup of X.*

Proof. We verify that $(\ker f)_{\leq p} = (\ker f')_p + (\ker f)_{\leq p-1}$, where $f' : X'_p \to Y'_{p+d}$ is the restriction of f. For this, we simply note that by definition, f induces a map $X'_p \oplus X_{\leq p-1} \to Y'_{p+d} \oplus Y_{\leq p+d-1}$ which preserves the decomposition.

The following lemma is straightforward to verify.

Lemma A.1.13. *Suppose $f : Y \to X$ is a degree d map of split-filtered Abelian groups. Then restricting the filtration on X to the image of f, we find* $(\mathrm{im}\, f)_{\leq p} = \mathrm{im}\left(f|_{Y_{\leq p-d}}\right)$. *Further,* $\mathrm{im}\, f' \subset \mathrm{im}\, f$ *defines a splitting, giving* $\mathrm{im}\, f$ *the structure of a split-filtered Abelian group.*

Lemma A.1.14. *Suppose $f : Y \to X$ is a degree d map of split-filtered Abelian groups. Then $\operatorname{coker}(f') \subset \operatorname{coker}(f)$ defines a splitting, giving $\operatorname{coker}(f)$ the structure of a split-filtered Abelian group.*

Proof. Via Lemma A.1.13, we know that $\operatorname{im}(f') \subset \operatorname{im}(f)$ defines a split-filtered structure on $\operatorname{im}(f)$. As $\operatorname{coker}(f) = \operatorname{coker}(\operatorname{im}(f) \to X)$, it therefore suffices to consider the case where f is injective. We have a diagram of (split) short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y'_p & \longrightarrow & Y_{\le p} & \longrightarrow & Y_{\le p-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y'_p & \longrightarrow & Y_{\le p} & \longrightarrow & Y_{\le p-1} & \longrightarrow & 0 \end{array}$$

where the vertical maps are injections. By the snake lemma, this gives a split short exact sequence of cokernels $(X/Y)_{\le p} = (X'/Y')_p \oplus (X/Y)_{\le p}$. In particular, the inclusion $X_p \to X_{\le p}$ induces an inclusion $(X'/Y')_p \subset (X/Y)_{\le p}$ giving our desired splitting. □

Proposition A.1.15. *The category of split-filtered Abelian groups is an Abelian category which is cocomplete, i.e. closed under colimits.*

Proof. The fact that we have an Abelian category is a consequence of Lemma A.1.14, Lemma A.1.13, Lemma A.1.12. By [Wei94, Prop. 2.6.8] cocompleteness follows from being closed under direct sums, which can be checked by noticing that $\bigoplus_{\lambda\in\Lambda} X^\lambda$ is split-filtered with respect to the graded subgroup $\bigoplus_{\lambda\in\Lambda}(X^\lambda)'$. □

A.2. Modules and Tensors.

Definition A.2.1. Let R be a ring and M a left (or right) R-module. We say that M is a split-filtered R module if it is a split-filtered Abelian group such that $M_{\le n}$, M', and M'_n are R-submodules of M for all n.

Lemma A.2.2. *Let R be a ring. Suppose M is a split-filtered right R-module and N a left R-module. Then the natural maps $M_{\le p} \otimes_R N \to M \otimes_R N$ and $M'_p \otimes_R N \to M \otimes_R N$ are injective.*

Proof. As $M \otimes_R N$ is a directed limit of $M_{\le i} \otimes_R N$ taken over all i, it follows that an element $x \in M_{\le p} \otimes_R N$ maps to 0 in $M \otimes_R N$ if and only if it maps to 0 in $M_{\le i} \otimes_R N$ for some $i > p$. Consequently, by induction, it suffices to show that the map $M_{\le i} \otimes_R N \to M_{\le i+1} \otimes_R N$ is injective for all i. But note that (for any i, not just $i > p$):

$$M_{\le i+1} \otimes_R N = (M_{\le i} \oplus M'_{i+1}) \otimes_R N \cong (M_{\le i} \otimes_R N) \oplus (M'_{i+1} \otimes_R N),$$

from which we see that $M_{\le i} \otimes_R N \to M_{\le i+1} \otimes_R N$ is in fact split injective and therefore $M_{\le p} \otimes_R N \to M \otimes_R N$ is injective.

But by this same reasoning, we see (taking $i+1 = p$) that $M'_p \otimes_R N \to M_{\le p} \otimes_R N$ is split injective. Since $M_{\le p} \otimes_R N \to M \otimes_R N$ is injective by the previous paragraph, we find that $M'_p \otimes_R N \to M \otimes_R N$ is injective as well. □

Definition/Lemma A.2.3. Let R be a ring, M a split-filtered right R-module and N a split-filtered left R-module. Then $M \otimes_R N$ is naturally a split-filtered R-module by defining $(M \otimes_R N)_{\le n}$ to be $\sum_{p+q=n} M_{\le p} \otimes_R N_{\le q}$ and $(M \otimes_R N)'_n$ to be $\bigoplus_{p+q=n} M'_p \otimes_R N'_q$.

Note that it follows from Lemma A.2.2 that these are in fact submodules of $M \otimes_R N$.

Proof. We have:

$$\begin{aligned}M_{\leq p} \otimes_R N_{\leq q} &= (M'_p \oplus M_{\leq p-1}) \otimes_R (N'_q \oplus N_{\leq q-1})\\ &= (M'_p \otimes_R N'_q) \oplus (M_{\leq p-1} \otimes_R N'_q) \oplus (M'_p \otimes_R N_{\leq q-1}) \oplus (M_{\leq p-1} \otimes_R N_{\leq q-1})\\ &\subseteq (M' \otimes_R N')_n \oplus (M \otimes_R N)_{\leq n-1}.\end{aligned}$$

This shows $(M \otimes_R N)_{\leq n} \subseteq (M' \otimes_R N')_n \oplus (M \otimes_R N)_{\leq n-1}$. The other inclusion is straightforward. □

A.3. Rings and Ideals.

Definition A.3.1. If U is a filtered Abelian group with a (not necessarily associative, not necessarily unital) ring structure, we say that it is a filtered ring if $U_{\leq p}U_{\leq q} \subset U_{\leq p+q}$. If U is a filtered ring and we are given U' a graded subring providing a splitting, we say U is a split-filtered ring.

Definition A.3.2. Let U be a split-filtered ring and M a left U-module, split-filtered as an Abelian group. We say that M is a split-filtered U-module if we have $U_{\leq p}M_{\leq q} \subset M_{\leq p+q}$ and $U'_pM'_q \subset M'_{p+q}$. Equivalently, M is a split-filtered U-module if the multiplication map $U \otimes_{\mathbb{Z}} M \to M$ is a split-filtered map (where the split filtration on $U \otimes_{\mathbb{Z}} M$ is described in Definition/Lemma A.2.3).

Definition/Lemma A.3.3. Let U be a split-filtered ring. M a split-filtered right U-module and N a split-filtered left U-module. Then $M \otimes_U N$ is naturally a split-filtered U-module by defining $(M \otimes_U N)_{\leq n}$ and $(M \otimes_U N)'_n$ to be the images in $M \otimes_U N$ of $\bigoplus_{p+q=n} M_{\leq p} \otimes_{\mathbb{Z}} N_{\leq q}$ and $\bigoplus_{p+q=n} M'_p \otimes_{\mathbb{Z}} N'_q$ respectively.

Proof. Consider the map $f\colon M \otimes_{\mathbb{Z}} U \otimes_{\mathbb{Z}} N \to M \otimes_{\mathbb{Z}} N$ given by $f(x \otimes u \otimes y) = xu \otimes y - x \otimes uy$. By definition, $M \otimes_U N$ is defined as the cokernel of this map. Regarding the domain and codomain as split-filtered via Definition/Lemma A.2.3, we see that this is a degree 0 map of split-filtered Abelian groups. So by Lemma A.1.14, the cokernel is split-filtered. □

Lemma A.3.4. *Let U be a split-filtered ring and let $S, T \subset U$ be arbitrary split-filtered additive subgroups. Then ST is split-filtered with $(ST)_{\leq n} = \sum_{p+q=n} S_{\leq p}T_{\leq q}$ and $(ST)'_n = \sum_{p+q=n} S'_pT'_q$.*

Proof. If we consider the tensor product $S \otimes_{\mathbb{Z}} T$, with its split-filtered structure of Definition/Lemma A.3.3, we see that the multiplication map $S \otimes_{\mathbb{Z}} T \to ST \subset U$ is a degree 0 map of split-filtered groups. The result now follows from Lemma A.1.13. □

Lemma A.3.5. *Let U be a split-filtered associative, unital ring, and let $X \subset U$ be a split-filtered additive subgroup. Then the ideal generated by X in U is also split-filtered with homogeneous part the ideal of U' generated by X'.*

Proof. It follows from Definition/Lemma A.2.3 that $U \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} U$ is split-filtered with homogeneous part $U' \otimes_{\mathbb{Z}} X' \otimes_{\mathbb{Z}} U'$. As the multiplication map $U \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} U \to U$ is a map of degree 0, it follows that its image, the ideal generated by X is split-filtered. □

Lemma A.3.6. *Suppose L is a split-filtered Lie algebra over a commutative (associative and unital) ring R. Then the universal enveloping algebra $U(L)$ is a split-filtered algebra with respect to the graded subalgebra $U(L') \subset U(L)$.*

Proof. It follows from Definition/Lemma A.3.3 and Proposition A.1.15 that the tensor algebra $T(L)$ is split-filtered with respect to $T(L')$. Let $X \subset T(L)$ be the image of the map $L \otimes_{\mathbb{Z}} L \to T(L)$ defined by $x \otimes y \mapsto x \otimes y - yx - [x, y]$ (note the tensor in the preimage is over $\mathbb{Z}$ and in the image is over R). As this is a map of degree 0, its image X is split-filtered with homogeneous part spanned by the analogous expressions with homogeneous elements. By Lemma A.1.13, it follows that the ideal generated by X is also split-filtered. Finally Lemma A.1.14 tells us that the quotient by this ideal, the universal enveloping algebra, is also split-filtered as described. □

A.4. Seminorms. The algebraic structures which naturally arise in studying the universal enveloping algebras of a VOA come with additional topological structure in the form of a seminorm. In this section we will examine seminorms and their interactions with gradings, filtrations and split filtrations.

Definition A.4.1. A system of neighborhoods of 0 in an Abelian group X is a collection of subgroups $\mathrm{N}^n X \subset X$, $n \in \mathbb{Z}$, with $\mathrm{N}^n X \subset \mathrm{N}^{n-1} X$ and $\bigcup_n \mathrm{N}^n X = X$.

Definition A.4.2. A (non-Archimedean) seminorm on an Abelian group X is a function $X \to \mathbb{R}_{\geq 0}$, $x \mapsto |x|$ such that $|0| = 0$ and $|x + y| \leq \max\{|x|, |y|\}$.

Definition A.4.3. A pseudometric on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Remark A.4.4. The notion of a system of neighborhoods is equivalent to the notion of an Abelian group seminorm (we always assume these to be non-Archimedean), where we would set $|x| = e^{-n}$ if $x \in \mathrm{N}^n X \setminus \mathrm{N}^{n+1} X$ or $|x| = 0$ if $x \in \bigcap_n \mathrm{N}^n X$. Such a seminorm also gives rise to a pseudometric by setting $d(x, y) = |x - y|$. Finally, these give rise to a topology on X whose basis is given by open balls with respect to this pseudometric. We see that addition is continuous with respect to this topology.

With this remark in mind, we will refer to systems of neighborhoods of 0 and seminorms interchangeably, and will often refer to an Abelian group with a system of neighborhoods of 0 as a seminormed Abelian group.

Remark A.4.5. It follows from the definition that a system of neighborhoods (and hence a seminorm) is precisely the same as an exhaustive right filtration. We may therefore consider the seminorm associated to either a right or left exhaustive filtration (in view of Remark A.1.2).

Definition A.4.6 *(Restriction of seminorms).* If X is a seminormed Abelian group and $Y \subset X$ is a subgroup, we will consider Y a seminormed Abelian group via the restriction of the seminorm. That is, we set $\mathrm{N}^n Y = \mathrm{N}^n X \cap Y$.

Definition A.4.7 *(Seminormed rings and modules).* Let U be a ring which is seminormed as an Abelian group. We say that U is a seminormed ring if $|xy| \leq |x||y|$, or, equivalently, $(\mathrm{N}^p U)(\mathrm{N}^q U) \subset \mathrm{N}^{p+q} U$. If M is a left U-module which is seminormed as an Abelian group, we say that it is a seminormed left module if $|xm| \leq |x||m|$ for all $x \in U, m \in M$.

Remark A.4.8. It follows immediately from Definition A.3.1 that a filtered ring (not necessarily associative or unital) becomes a seminormed ring with respect to the seminorm induced by the filtration as in Remark A.4.5, and that multiplication map is continuous with respect to the induced topology.

Warning A.4.9. We will often consider seminorms on rings which are not ring seminorms, but just Abelian group seminorms.

Definition A.4.10 *(Split-filtered seminorms).* Let X be a split-filtered Abelian group. We say that a seminorm is split-filtered if each of its neighborhoods $\mathrm{N}^n X$ are split-filtered subgroups of X. In this case we will simply refer to X as a split-filtered seminormed Abelian group.

The following notion captures a property that we will often seek: that smaller filtered parts of a given filtered Abelian group lie in progressively smaller neighborhoods.

Definition A.4.11 *(Tight seminorms).* Suppose X is a filtered seminormed Abelian group. We say that the X is tightly seminormed if for all m, p there exists d such that $X_{\leq -d} \subset \mathrm{N}^m X_{\leq p}$.

Lemma A.4.12. *Suppose X is a split-filtered seminormed Abelian group whose seminorm is tight. Then X' is dense in X.*

Proof. Let $x \in X$. We can choose n, p with $x \in \mathrm{N}^n X_{\leq p} \subset X$. For any m, we need to show that there exists $x' \in X'$ with $x - x' \in \mathrm{N}^m X$. We can write $\mathrm{N}^n X_{\leq p} = \mathrm{N}^n X'_p \oplus \mathrm{N}^n X_{\leq p-1}$ and iterating this expression, we find $\mathrm{N}^n X_{\leq p} = \bigoplus_{i=p-d+1}^{p} \mathrm{N}^n X'_i \oplus \mathrm{N}^n X_{\leq p-d}$ for any $d > 0$. But by the tightness of the seminorm, choosing $d \gg 0$ we can ensure $\mathrm{N}^n X_{\leq p-d} \subset X_{\leq p-d} \subset \mathrm{N}^m X_{\leq p}$. In particular, we may write $x = x' + y$ with $x' \in \bigoplus_{i=p-d+1}^{p} \mathrm{N}^n X'_i \subset X'$ and $y \in \mathrm{N}^m X_{\leq p}$ as desired. □

A.5. Graded and Filtered Completions.

Definition A.5.1 *(Graded-complete and filtered-complete Abelian groups).* Let X be a normed Abelian group. If X is graded, we say that it is graded-complete if each of the graded subspaces X_n is complete. If X is filtered, we say that is is filtered-complete if each subspace $X_{\leq n}$ is complete.

Definition A.5.2 *(Short homomorphisms).* Let X, Y be seminormed Abelian groups. A group homomomorphism $f : X \to Y$ is called a short (or metric) homomorphism if $|f(x)| \leq |x|$ for all $x \in X$.

Definition/Lemma A.5.3 *(Separated completions).* Let X be a seminormed Abelian group. Then we may form the (separated) completion $\widehat{X}$ of X, which is a complete normed Abelian group equipped with a short map $\iota : X \to \widehat{X}$ which is universal for short maps to complete normed Abelian groups. That is, for every complete normed Abelian group Y and short homomorphism $X \to Y$, there is a unique factorization of this map as $X \xrightarrow{\iota} \widehat{X} \to Y$.

This can be constructed in the usual way via equivalence classes of Cauchy sequences. The following Lemma is a consequence of the fact that a metric space maps injectively into its completion:

Lemma A.5.4. *Let X be a seminormed Abelian group. Then the canonical map $X \to \widehat{X}$ has kernel $\bigcap_{n \in \mathbb{Z}} \mathrm{N}^n X$. In particular, $X \to \widehat{X}$ is injective exactly when the seminorm on X is actually a norm.*

Lemma A.5.5. *Let $W \subset Z \subset X$ be subgroups of a seminormed Abelian group X. Then in the induced seminorm on Z/W, the separated completion of Z/W can be identified with $\widehat{Z}/\widehat{W}$, and $\widehat{Z}$, $\widehat{W}$ can be identified with the closures of the images of Z and W in $\widehat{X}$ respectively.*

Proof. The latter identification of completions and closures is straightforward to check. We note that there is a universal map $\widehat{Z} \to \widehat{(Z/W)}$ of separated completions with W in the kernel. But as the image is Hausdorff, it follows that $\overline{W}$ must also be in the kernel. But now we see that the map $Z/W \to \widehat{Z}/\widehat{W}$ is therefore universal giving us $\widehat{Z}/\widehat{W} \cong \widehat{(Z/W)}$ as desired. □

We have various closely related universal constructions as follows.

Definition/Lemma A.5.6 (Filtered and graded completions). Let X be a seminormed Abelian group. If X is graded, then we can construct a short homomorphism $X \to \widehat{X}^{\mathsf{g}}$ which is universal for short homomorphisms to graded-complete Abelian groups. If X is filtered, then we can construct a short homomorphism $X \to \widehat{X}^{\mathsf{f}}$ which is universal for short homomorphisms to filtered-complete Abelian groups.

Proof. We set $\widehat{X}^{\mathsf{g}} = \bigoplus_n \widehat{X}^{\mathsf{g}}_n$ where $\widehat{X}^{\mathsf{g}}_n = \widehat{X_n}$, and $\widehat{X}^{\mathsf{f}} = \bigcup_n \widehat{X}^{\mathsf{f}}_{\le n}$ where $\widehat{X}^{\mathsf{f}}_{\le n} = \widehat{X_{\le n}}$. □

Remark A.5.7. These are also described in [MNT10] as the degreewise completion and the filterwise completion respectively.

Lemma A.5.8. *For X a split-filtered tightly seminormed Abelian group, $\widehat{X}^{f}$ is a split-filtered tightly seminormed Abelian group with respect to the graded subgroup $\widehat{X'}^{g}$.*

Proof. Let us note that the natural morphism $\widehat{X'}^{\mathsf{g}} \to \widehat{X}^{\mathsf{f}}$ is an inclusion. For this, suppose that we have a pair of Cauchy sequences (a_n), (b_n) in X'_p which have the same image in $\widehat{X}^{\mathsf{f}}$. Without loss of generality, we may select subsequences and re-index (possibly after modifying our starting index to an appropriate integer), and assume $a_n - b_n \in \mathrm{N}^n X_{\le p}$ for all n. But $\mathrm{N}^n X'_p = \mathrm{N}^n X_{\le p} \cap X'_p$ tells us that these Cauchy sequences have the same limit in $\widehat{X'}^{\mathsf{g}}$ as well, giving injectivity.

Next we check that $\widehat{X}^{\mathsf{f}}_{\le p-1} \cap \widehat{X'}^{\mathsf{g}}_p = 0$. Suppose we have an equality of classes of Cauchy sequences $(x_n) = (y_n)$ where $x_n \in X_{\le p-1}$, $y_n \in X'_p$ and $x_n - y_n \in \mathrm{N}^n X_{\le p}$. We claim that we may replace (x_n) by an equivalent Cauchy sequence (x'_n) with $x'_n \in X_{\le p-d}$ for all $d > 0$. To see this, we argue by induction on d. Suppose $x_n \in X_{\le p-(d-1)}$ we use the fact that our seminorm is split-filtered to write $x_n = x'_n + x''_n$ with $x'_n \in X_{\le p-d}$ and $x''_n \in X_{p-(d-1)}$. As $y_n - x_n = y_n - x'_n - x''_n \in \mathrm{N}^n X_{\le p}$ and

$$\mathrm{N}^n X_{\le p} = \mathrm{N}^n X_p + \mathrm{N}^n X_{\le p-1} = \cdots = \mathrm{N}^n X_p \oplus \mathrm{N}^n X_{p-1} \oplus \cdots \oplus \mathrm{N}^n X_{p-d+1} \oplus \mathrm{N}^n X_{\le p-d},$$

we see that an element of $X_{\le p}$ lies in the n'th neighborhood $\mathrm{N}^n X_{\le p}$ if and only if each of its factors with respect to the decomposition

$$X_{\le p} = X_p \oplus X_{p-1} \oplus \cdots \oplus X_{p-d+1} \oplus X_{\le p-d},$$

lies in their corresponding n'th neighborhood. We therefore find $x''_n \in \mathrm{N}^n X_{p-d+1}$ for all n. Consequently $\lim_{n\to\infty} x''_n = 0$, which says the Cauchy sequences (x_n) and (x'_n) are equivalent. This verifies our claim.

Now, we claim that $(y_n) = 0$. By definition of the completion, this amounts to $(y_n) \in \mathrm{N}^m X_{\leq p-1}$ for all m. By our hypothesis, for any m there exists d' such that $X_{\leq -d'} \subset \mathrm{N}^m X_{\leq p-1}$. In particular, choosing $d = d' + p$ in the prior argument, we find that we may choose a Cauchy sequence (y'_n) equivalent to the first, with $y'_n \in X_{\leq -d'} \subset \mathrm{N}^m X_{\leq p-1}$, showing that $(y'_n) \in \mathrm{N}^m \widehat{X}^{\mathsf{f}}_{\leq p-1}$ for all m, verifying our claim.

Now we check $\mathrm{N}^n \widehat{X}^{\mathsf{f}}_{\leq p} \subset \mathrm{N}^n \widehat{X}^{\mathsf{f}}_{\leq p-1} + \mathrm{N}^n \widehat{X'}^{\mathsf{g}}_p$. For this, let $\sum x_i \in \mathrm{N}^n \widehat{X}^{\mathsf{f}}_{\leq p}$ be a convergent infinite series. Without loss of generality, we may assume $x_i \in \mathrm{N}^{n+i} X_{\leq p}$ for all i. As our seminorm is split-filtered, we can write $x_i = (x_i)_p + (x_i)_{<p}$ as in Notation A.1.7. But now we see that the sums $\sum_i (x_i)_p$ and $\sum_i (x_i)_{<p}$ both converge in $\mathrm{N}^n \widehat{X'}^{\mathsf{g}}_p$ and $\mathrm{N}^n \widehat{X}^{\mathsf{f}}_{\leq p-1}$ respectively, showing that $\sum x_i \in \mathrm{N}^n \widehat{X}^{\mathsf{f}}_{\leq p-1} + \mathrm{N}^n \widehat{X'}^{\mathsf{g}}_p$ as desired.

As $\widehat{X}^{\mathsf{f}} = \bigcup_n \mathrm{N}^n \widehat{X}^{\mathsf{f}}$ and $\widehat{X'}^{\mathsf{g}} = \bigcup_n \mathrm{N}^n \widehat{X'}^{\mathsf{g}}$ we conclude that X is split-filtered seminormed by Proposition A.1.15.

To check that it is tightly seminormed, We notice that whenever $X_{\leq -d} \subset \mathrm{N}^m X_{\leq p}$ we find that, upon taking closures in $\widehat{X}^{\mathsf{f}}_{\leq p}$, that $\widehat{X}^{\mathsf{f}}_{\leq -d} \subset \mathrm{N}^m \widehat{X}^{\mathsf{f}}_{\leq p}$, showing that $\widehat{X}^{\mathsf{f}}$ is also tightly seminormed (with the same choice of d for a given m, p). □

Remark A.5.9. In light of this result, it makes sense to refer to $\widehat{X}^{\mathsf{f}}$ as the completion of X, when X is split-filtered, with the understanding that the graded subgroup is given by $\widehat{X'}^{\mathsf{g}}$. In the case $X = \widehat{X}^{\mathsf{f}}$, we say X is complete.

Definition/Lemma A.5.10. Suppose X is a split-filtered, tightly seminormed, complete Abelian group, and suppose $Y \subset X$ is a split-filtered subgroup. Define the closure $\overline{Y}$ of Y in X to be the filtered subgroup with $\overline{Y}_{\leq p}$ the closure of the image of $Y_{\leq p}$ in $X_{\leq p}$ and $\overline{Y'}_p$ the closure of the image of Y'_p in X'_p. Then $\overline{Y}$ is split-filtered with respect to the graded subgroup $\overline{Y}'$.

Proof. It follows from the definition that the restriction of a tight seminorm is again a tight seminorm. As we can identify the closures with the completions by Lemma A.5.5, the result follows from Lemma A.5.8. □

A.6. Canonical Seminorms. The seminorms used in studying universal enveloping algebras of VOAs arise in a very specific way, as described in [TUY89,FZ92,FBZ04, Fre07,NT05,MNT10]. We will recall a generalized definition of these seminorms, as in [MNT10], and then examine some abstract features in the context of split filtrations, which will allow us to relate the filtered and graded versions.

Definition A.6.1 *(The canonical seminorm).* Let U be a filtered ring. The canonical system of neighborhoods on U is defined by ${}^c\mathrm{N}^n U = U U_{\leq -n}$ (a left ideal of U if U is associative). We will write ${}^c|\cdot|$ for the corresponding canonical seminorm.

Lemma A.6.2. *Suppose $U' \subset U$ is a split-filtered ring. Then the canonical seminorm is split-filtered and tight.*

Proof. Suppose $u \in {}^c\mathrm{N}^n U_{\leq p}$. By Lemma A.3.4, we can write u as a sum of elements of the form $\alpha\beta$ with $\alpha \in U_{\leq a}$ and $\beta \in U_{\leq b}$ with $a + b = p$ and $b \leq -n$.

Using our splitting we may write $\alpha = \overline{\alpha} + \alpha'$ and $\beta = \overline{\beta} + \beta'$ with $\overline{\alpha} \in U'_a, \alpha' \in U_{\leq a-1}$, $\overline{\beta} \in U'_b$, $\beta' \in U_{\leq b-1}$, and so we have $\alpha'\overline{\beta} \in {}^c\mathrm{N}^n U_{\leq p-1}$ giving us:

$$\alpha\beta = \overline{\alpha}\overline{\beta} + \overline{\alpha}\beta' + \alpha'\overline{\beta} + \alpha'\beta' \in {}^c\mathrm{N}^n U'_p + {}^c\mathrm{N}^n U_{\leq p-1} + {}^c\mathrm{N}^{n+1} U_{\leq p-1} = {}^c\mathrm{N}^n U'_p + {}^c\mathrm{N}^n U_{\leq p-1}.$$

It follows that ${}^c\mathrm{N}^n U_{\le p} \subset {}^c\mathrm{N}^n U_{\le p-1} + {}^c\mathrm{N}^n U'_p$, and hence ${}^c\mathrm{N}^n U_{\le p} = {}^c\mathrm{N}^n U_{\le p-1} + {}^c\mathrm{N}^n U'_p$, showing the canonical seminorm is split-filtered.

To check that it is tight, we simply notice that for any m, p, we have for $d \ge \max\{m, -p\}$, $U_{\le -d} \subset U_{\le p} \cap {}^c\mathrm{N}^m U = {}^c\mathrm{N}^m U_{\le p}$. □

The following Lemmas are easily verified.

Lemma A.6.3. *Let U be a filtered associative ring. Then for any $p, q, n \in \mathbb{Z}$, we have*

$$\left({}^c\mathrm{N}^n U_{\le p}\right) U_{\le q} \subset {}^c\mathrm{N}^{n-q} U_{\le p+q} \quad \textit{and} \quad U_{\le p}\left({}^c\mathrm{N}^n U_{\le q}\right) \subset {}^c\mathrm{N}^n U_{\le p+q}.$$

Lemma A.6.4. *Let $f: X \to Y$ be a surjective filtered homomorphism of filtered associative rings. Then $f({}^c\mathrm{N}^n X) = {}^c\mathrm{N}^n Y$.*

By Lemma A.6.5, a useful property of the canonical topology is that multiplication is continuous with respect to it, at least when restricted to the various filtered parts.

Lemma A.6.5. *Let U be a filtered associative ring equipped with a seminorm such that*

$$\left(\mathrm{N}^n U_{\le p}\right) U_{\le q} \subset \mathrm{N}^{n-q} U_{\le p+q} \quad \textit{and} \quad U_{\le p}\left(\mathrm{N}^n U_{\le q}\right) \subset \mathrm{N}^n U_{\le p+q}.$$

Then for any p, q, the multiplication map $U_{\le p} \times U_{\le q} \to U_{\le p+q}$ is continuous with respect to the seminorm in both variables. Consequently, the completion $\widehat{U}^{\mathsf{f}}$ naturally has the structure of an associative ring.

Remark A.6.6. It follows that under these hypotheses, if U and its seminorm is split-filtered, then the multiplication map $U'_p \times U'_q \to U'_{p+q}$ is also continuous (being the restriction of a continuous map). Consequently, in this case, the completion $\widehat{U}^{\mathsf{f}}$ is also a split-filtered associative ring, which is tightly split-filtered if U is (by Lemma A.5.8).

Proof. Let $u_1 \in U_{\le p}, u_2 \in U_{\le q}$. Then we must show that multiplication is continuous with respect to both variables at (u_1, u_2). That is, given $d \in \mathbb{Z}$, we must show there exist n_1, n_2 such that $(u_1 + \mathrm{N}^{n_1} U_{\le p})u_2 \subset u_1u_2 + \mathrm{N}^d U_{\le p+q}$ and $u_1(u_2 + \mathrm{N}^{n_2} U_{\le q}) \subset u_1u_2 + \mathrm{N}^d U_{\le p+q}$. By our hypotheses, for $n_2 \ge d$, we have $u_1(u_2 + \mathrm{N}^{n_2} U_{\le q}) \subset u_1u_2 + \mathrm{N}^{n_2} U_{\le p+q} \subset u_1u_2 + \mathrm{N}^d U_{\le p+q}$. On the other hand, for $n_1 \ge q+d$, we find $(\mathrm{N}^{n_1} U_{\le p})u_2 \subset \left(\mathrm{N}^{n_1} U_{\le p}\right) U_{\le q} \subset \mathrm{N}^{n_1-q} U_{\le p+q} \subset \mathrm{N}^d U_{\le p+q}$, as desired. □

Remark A.6.7. If U is a split-filtered seminormed ring with ${}^c\mathrm{N}^n U_{\le p} \subset \mathrm{N}^n U_{\le p}$ then by Lemma A.6.2, it is tightly seminormed.

The canonical seminorm on a split-filtered associative ring has a number of useful properties which we would like to axiomatize. As we have seen, it is tight and split-filtered (Lemma A.6.2) and verifies the identities of Lemma A.6.3.

Definition A.6.8. Let U be a split-filtered seminormed associative ring. We say the seminorm is almost canonical if it verifies the following conditions:

(a) the seminorm is split-filtered,
(b) $\mathrm{N}^n U_{\le p} = {}^c\mathrm{N}^n U_{\le p} + \mathrm{N}^{n+1} U_{\le p}$ for all n, p,
(c) $\left(\mathrm{N}^n U_{\le p}\right) U_{\le q} \subset \mathrm{N}^{n-q} U_{\le p+q}$ and $U_{\le p}\left(\mathrm{N}^n U_{\le q}\right) \subset \mathrm{N}^n U_{\le p+q}$ for all p, q, n.

Lemma A.6.9. *Let U be a split-filtered almost canonically seminormed associative ring. Then $\mathrm{N}^n U'_p = {}^c\mathrm{N}^n U'_p + \mathrm{N}^{n+1} U'_p$ for all n, p.*

Proof. Using the fact that the seminorm is split-filtered and Definition A.6.8 (b), we have $\mathrm{N}^n U'_p + \mathrm{N}^n U_{\leq p-1} = {}^c\mathrm{N}^n U'_p + {}^c\mathrm{N}^n U_{\leq p-1} + \mathrm{N}^{n+1} U'_p + \mathrm{N}^{n+1} U_{\leq p-1}$ from which the result follows looking modulo $U_{\leq p-1}$. □

Lemma A.6.10. *Let U be a split-filtered seminormed associative ring. Then the following are equivalent:*

(a) $\mathrm{N}^n U_{\leq p} = {}^c\mathrm{N}^n U_{\leq p} + \mathrm{N}^{n+1} U_{\leq p}$ *for all* n, p,
(b) $\mathrm{N}^n U_{\leq p} = {}^c\mathrm{N}^n U_{\leq p} + \mathrm{N}^{n+d} U_{\leq p}$ *for all* n, p *and* $d > 0$,
(c) ${}^c\mathrm{N}^n U_{\leq p}$ *is contained in and dense in* $\mathrm{N}^n U_{\leq p}$.

Proof. This follows by iterating the expression in part (a). □

Remark A.6.11. In particular, if U carries an almost canonical seminorm, then by Definition A.6.8(b), it satisfies the equivalent conditions of Lemma A.6.10, and from Lemma A.6.10(c), it follows from Remark A.6.7 that its seminorm is also tight.

Lemma A.6.12. *Let $f: X \to Y$ be a surjective map of split-filtered associative rings, and suppose X is endowed with an almost canonical seminorm. Then the system of neighborhoods $\mathrm{N}^n Y_{\leq p} = f(\mathrm{N}^n X_{\leq p})$ defines an almost canonical seminorm on Y.*

Proof. By Lemma A.6.4 the image of the canonical neighborhoods in X are canonical neighborhoods in Y, and by definition the images of neighborhoods in X are neighborhoods in Y. The result then follows directly by applying the homomorphism f to the properties of Definition A.6.8 (b) and (c). □

Lemma A.6.13. *Suppose U is a split-filtered associative ring with an almost canonical seminorm. Then the induced norm on the filtered completion $\widehat{U}^{\mathsf{f}}$ is also almost canonical.*

Proof. By Remark A.6.7 and Lemma A.5.8, $\widehat{U}^{\mathsf{f}}$ is split-filtered and tightly seminormed, implying that $\widehat{U}^{\mathsf{f}}$ satisfies Definition A.6.8 (a). We proceed to Definition A.6.8 (b) using the equivalent conditions of Lemma A.6.10. As the neighborhoods $\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$ can be identified as the closure of the image of $\mathrm{N}^n U_{\leq p}$ and ${}^c\mathrm{N}^n U_{\leq p}$ is dense in $\mathrm{N}^n U_{\leq p}$, it follows that the image of ${}^c\mathrm{N}^n U_{\leq p}$ is dense in $\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$. But as the image of ${}^c\mathrm{N}^n U_{\leq p}$ is contained in ${}^c\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$, it follows that ${}^c\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$ is also dense in $\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$, verifying Definition A.6.8 (b).

We verify the first part of Definition A.6.8 (c) (the second part is analogous). The multiplication map $\mathrm{N}^n U_{\leq p} \times U_{\leq q} \to U_{\leq p+q}$ is continuous by Lemma A.6.5 and it factors through $\mathrm{N}^{n-q} U_{\leq p+q}$. By continuity, taking closures (of the images) in the completions $\widehat{U}^{\mathsf{f}}_{\leq p}, \widehat{U}^{\mathsf{f}}_{\leq q}, \widehat{U}^{\mathsf{f}}_{\leq p+q}$ of $U_{\leq p}, U_{\leq q}, U_{\leq p+q}$ respectively, we find that our map extends to a continuous map $\overline{\mathrm{N}^n U_{\leq p}} \times \overline{U_{\leq q}} \to \overline{U_{\leq p+q}}$ which factors through $\overline{\mathrm{N}^{n-q} U_{\leq p+q}}$. Since the closure of the image in a completion can be identified with the completion itself, and $\overline{\mathrm{N}^n U_{\leq p}} = \mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p}$, $\overline{\mathrm{N}^{n-q} U_{\leq p+q}} = \mathrm{N}^{n-q} \widehat{U}^{\mathsf{f}}_{\leq p+q}$, we interpret our multiplication as a continuous map $\mathrm{N}^n \widehat{U}^{\mathsf{f}}_{\leq p} \times \widehat{U}^{\mathsf{f}}_{\leq q} \to \widehat{U}^{\mathsf{f}}_{\leq p+q}$ which factors through $\mathrm{N}^{n-q} \widehat{U}^{\mathsf{f}}_{\leq p+q}$, as desired. □

A.7. Completed Tensors. Completed tensors, introduced here in Definition A.7.2, make a number of arguments more natural.

Definition A.7.1 (Seminorm on tensors). Let R be a seminormed ring, M a right seminormed R-module and N a left seminormed R-module. We define a seminorm on $M \otimes_R N$ by the following neighborhoods of 0:

$$\mathrm{N}^n(M \otimes_R N) = \sum_{p+q=n} \mathrm{im}\left((\mathrm{N}^p M \otimes_{\mathrm{N}^0 R} \mathrm{N}^q N) \to M \otimes_R N \right).$$

Definition A.7.2 (Complete tensors). Let R be a seminormed ring, M a right seminormed R-module, and N a left seminormed R-module. The complete tensor product $M\widehat{\otimes}_R N$ is defined to be the completion of the seminormed Abelian group $M \otimes_R N$ with seminorm as described in Definition A.7.1.

Definition/Lemma A.7.3 (Complete tensors, filtered and graded). Let R be a seminormed ring, M a right seminormed R-module and N a left seminormed R-module. If R, M, N are graded then we can construct a short homomorphism $M \times N \to M\widehat{\otimes}^{\mathsf{g}}_R N$ which is universal for R-bilinear maps to graded-complete Abelian groups. If R, M, N are filtered then we can construct a short homomorphism $M \times N \to M\widehat{\otimes}^{\mathsf{f}}_R N$ which is universal for R-bilinear maps to filtered-complete Abelian groups.

Proof. These are $M\widehat{\otimes}^{\mathsf{g}}_R N = \widehat{M \otimes_R N}^{\mathsf{g}}$ and $M\widehat{\otimes}^{\mathsf{f}}_R N = \widehat{M \otimes_R N}^{\mathsf{f}}$ respectively. □

A.8. Discrete Quotients. This section will be particularly useful in construction of generalized Verma modules and new algebraic structures (the mode transition algebras of Sect. 3.2) which will play an important role for us.

If U is a filtered ring, then $U_{\leq 0}$ is always a subring and $U_{\leq -n}$ for $n > 0$ is a two-sided ideal of $U_{\leq 0}$. Moreover, for $n > 0$, we have $U \otimes_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n} = U/UU_{\leq -n} = U/{}^c\mathrm{N}^n U$.

Lemma A.8.1. *Suppose U is a split-filtered almost canonically seminormed ring. Then*

$$U\widehat{\otimes}^{f}_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n} \cong \widehat{U}^{f}/\mathrm{N}^n\widehat{U}^{f} \cong \widehat{U'}^{g}/\mathrm{N}^n\widehat{U'}^{g} \cong U'\widehat{\otimes}^{g}_{U'_{\leq 0}} U'_{\leq 0}/U'_{\leq -n}$$

with isomorphism induced by the continuous map $U \otimes_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n} \to U/\mathrm{N}^n U$ via $u \otimes \overline{a} \mapsto \overline{ua}$.

In particular, as a topological space, these have the discrete topology.

Proof. It is immediate that, assuming the claimed equalities hold, the natural quotient topology on $\widehat{U'}^{\mathsf{g}}/\mathrm{N}^n\widehat{U'}^{\mathsf{g}}$ is discrete.

As we have noticed, $U \otimes_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n} \cong U/{}^c\mathrm{N}^n U$. We can therefore identify the separated completion of $U_{\leq p} \otimes_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n}$ with the separated completion of $U_{\leq p}/{}^c\mathrm{N}^n U_{\leq p}$. But since $\overline{{}^c\mathrm{N}^n U_{\leq p}} = \mathrm{N}^n U_{\leq p}$ by Lemma A.6.10, the isomorphism $U\widehat{\otimes}^{\mathsf{f}}_{U_{\leq 0}} U_{\leq 0}/U_{\leq -n} \cong \widehat{U}^{\mathsf{f}}/\mathrm{N}^n\widehat{U}^{\mathsf{f}}$ follows by Lemma A.5.5.

Next, we note that the natural map $\widehat{U'}^{\mathsf{g}} \to \widehat{U}^{\mathsf{f}}/\mathrm{N}^n\widehat{U}^{\mathsf{f}}$ has kernel $\mathrm{N}^n\widehat{U'}^{\mathsf{g}}$. As $U'_{\leq -m} \subset {}^c\mathrm{N}^n U \subset \mathrm{N}^n U$ for $m \gg 0$ it follows that our map $U' \to \widehat{U}^{\mathsf{f}}/\mathrm{N}^n\widehat{U}^{\mathsf{f}}$, which has dense image by Lemma A.4.12 factors through the surjection $U'/U'_{\leq -m} \to U'/\mathrm{N}^n U'$. In particular, the restriction of this map to $U'_{\leq p}/U'_{\leq -m}$ factors through $\widehat{U'_{\leq p}/U'_{\leq -m}}$ and hence the image of this part coincides with the image of $\widehat{U'_{\leq p}}^{\mathsf{g}}$. But $\widehat{U'_{\leq p}/U'_{\leq -m}} = \bigoplus_{-m<i\leq p} \widehat{U}'_i = \bigoplus_{-m<i\leq p} \widehat{U'}^{\mathsf{g}}_i$. We therefore find that the map $\widehat{U'}^{\mathsf{g}}_{\leq p} \to \widehat{U}^{\mathsf{f}}_{\leq p}/\mathrm{N}^n\widehat{U}^{\mathsf{f}}_{\leq p}$ factors through $\bigoplus_{-m<i\leq p} \widehat{U'}^{\mathsf{g}}_i$ which is a complete space. As this map has dense image, it is surjective and from our prior description of the kernel, we see $\widehat{U'}^{\mathsf{g}}_{\leq p}/\mathrm{N}^n\widehat{U'}^{\mathsf{g}}_{\leq p} \cong \widehat{U}^{\mathsf{f}}_{\leq p}/\mathrm{N}^n\widehat{U}^{\mathsf{f}}_{\leq p}$. Taking a union over all p gives the identification $\widehat{U}^{\mathsf{f}} \cong \widehat{U'}^{\mathsf{g}}/\mathrm{N}^n\widehat{U'}^{\mathsf{g}}$.

Making the same observations as in the beginning of the proof with U' instead of U, we may identify the separated completion of $U'_{\leq p} \otimes_{U'_{\leq 0}} U'_{\leq 0}/U'_{\leq -n}$ with the separated completion of $U'_{\leq p}/{}^c\mathrm{N}^n U'_{\leq p}$. Choosing m as in the previous paragraph, we find that we have a surjective map $U'/U'_{\leq -m} \to U'/\mathrm{N}^n U'$ which allows us to identify the separated completion of $U'_{\leq p}/{}^c\mathrm{N}^n U'_{\leq p}$ with $\widehat{U'}^{\mathsf{g}}_{\leq p}/\mathrm{N}^n \widehat{U'}^{\mathsf{g}}_{\leq p}$ as desired. □

A.9. Triples of Associative Algebras. In this section we collect some of our previous facts which will be useful for the construction of our universal enveloping algebras of a VOA. As we simultaneously construct and relate three versions of the enveloping algebra (left, right and finite Definition 2.4.3), we will therefore introduce notions for working with triples of associative algebras here.

Definition A.9.1. A good triple of associative algebras $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ consists of the data of a left split-filtered associative algebra U^{L}, a right split-filtered associative algebra U^{R}, and a graded subalgebra U' of both U^{L} and U^{R}.

Definition A.9.2. A morphism of good triples $(X^{\mathsf{L}}, X', X^{\mathsf{R}}) \to (Y^{\mathsf{L}}, Y', Y^{\mathsf{R}})$ is a pair of degree 0 maps of split-filtered associative algebras $X^{\mathsf{L}} \to Y^{\mathsf{L}}$ and $X^{\mathsf{R}} \to Y^{\mathsf{R}}$ which agree on $X' \to Y'$.

Definition A.9.3. A good seminorm on a good triple of associative algebras $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ consists of almost canonical split-filtered seminorms on U^{L} and U^{R} defined by neighborhoods $\mathrm{N}^n_{\mathsf{L}} U^{\mathsf{L}}$ and $\mathrm{N}^n_{\mathsf{R}} U^{\mathsf{R}}$ respectively such that $\mathrm{N}^n_{\mathsf{L}} U'_p = \mathrm{N}^{n+p}_{\mathsf{R}} U'_p$.

Remark A.9.4. We note that in the case $p = 0$ we have $\mathrm{N}^n_{\mathsf{L}} U'_0 = \mathrm{N}^n_{\mathsf{R}} U'_0$, and in this case we can unambiguously write $\mathrm{N}^n U'_0$ for each. Also in this case, it follows from Definition A.6.8 (c) that $\mathrm{N}^n U'_0$ is a two sided ideal of U'_0.

Lemma A.9.5. *Suppose $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ is a good triple of associative unital algebras and $I \lhd U'$ is a homogeneous ideal. Let $I^{\mathsf{L}} = U^{\mathsf{L}} I U^{\mathsf{L}}$ and $I^{\mathsf{R}} = U^{\mathsf{R}} I U^{\mathsf{R}}$ be the ideals of U^{L} and U^{R} generated by I. Then $(I^{\mathsf{L}}, I, I^{\mathsf{R}})$ is a good triple (of ideals).*

Proof. We note that the triple (I, I, I) is good, where we regard I itself as left and right filtered as in Example A.1.9. The result now follows from Lemma A.3.5, in light of the observation that the ideal generated by I in U' is I itself. □

The following Lemma is an immediate consequence of Definition/Lemma A.5.10 (note that good seminorms give rise to almost canonical ones which are tight by Remark A.6.11).

Lemma A.9.6. *Let $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ be a good triple of associative unital algebras and $(I^{\mathsf{L}}, I, I^{\mathsf{R}})$ a good triple of ideals. Then the closures $(\overline{I}^{\mathsf{L}}, \overline{I}, \overline{I}^{\mathsf{R}})$ is a good triple of ideals.*

Remark A.9.7. If our seminorms on a triple $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ are canonical, they are easily verified to be good: it is split-filtered by Lemma A.6.2 and satisfies the other conditions of Definition A.6.8 by definition of the canonical seminorm and by Lemma A.6.3.

Definition A.9.8. If $(U^{\mathsf{L}}, U', U^{\mathsf{R}})$ is a good triple with a good seminorm, its completion is the triple $\left(\widehat{U^{\mathsf{L}}}^{\mathsf{f}}, \widehat{U'}^{\mathsf{g}}, \widehat{U^{\mathsf{R}}}^{\mathsf{f}}\right)$. We say that a triple is complete if it is the same as its completion under the canonical map.

The following two results show that, in the appropriate sense, the class of good triples with good seminorms are closed under completions and homomorphic images.

Corollary A.9.9. *If* $(X^{\mathsf{L}}, X', X^{\mathsf{R}}) \to (Y^{\mathsf{L}}, Y', Y^{\mathsf{R}})$ *is a surjective map of good triples, and* $(X^{\mathsf{L}}, X', X^{\mathsf{R}})$ *has a good seminorm, the induced seminorm on* $(Y^{\mathsf{L}}, Y', Y^{\mathsf{R}})$ *is good.*

Proof. This is an immediate consequence of Lemma A.1.14 and Lemma A.6.12. □

Corollary A.9.10. *Good triples with good seminorms are closed under the operation of completion.*

Proof. This is an immediate consequence of Lemma A.6.13. □

Appendix B. Generalized Verma Modules and Mode Transition Algebras

In this section, our basic object will be a graded seminormed algebra. While such an algebra may come as part of a triple as described in the previous section, the graded structure will play the decisive role here. We will, however, occasionally regard our graded algebra as also (split-)filtered as in Example A.1.9.

B.1. Generalized Higher Zhu Algebras and Verma Modules.

Definition B.1.1. For a graded, seminormed unital algebra U, we define the *generalized n-th Zhu algebra* as $\mathsf{A}_n(U) = U_0/\, \mathrm{N}^{n+1} U_0$.

For $\alpha \in U_0$, we write $[\alpha]_n$ to denote the image of α in $\mathsf{A}_n(U)$, and write $[\alpha]$ if n is understood. Observe that $\mathsf{A}_n(U) = 0$ if $n \leq -1$ since $\mathrm{N}^i U_0 = U_0$ whenever $i \leq 0$.

Definition B.1.2. If U is a graded algebra with an almost canonical seminorm and W_0 is a left $\mathsf{A}_n(U)$-module, we define a U-module $\Phi^{\mathsf{L}}_n(W_0)$ by

$$\Phi^{\mathsf{L}}_n(W_0) = \left(U/\, \mathrm{N}^{n+1}_{\mathsf{L}} U\right) \otimes_{U_0} W_0 = \left(U/\, \mathrm{N}^{n+1}_{\mathsf{L}} U\right) \otimes_{\mathsf{A}_n(U)} W_0.$$

We will generally write $\Phi^{\mathsf{L}}(W_0)$ for $\Phi^{\mathsf{L}}_0(W_0)$. Following [FZ92,DLM98] we define:

Definition B.1.3. If U is a graded algebra with an almost canonical seminorm and W is a left U-module, we define an $\mathsf{A}_n(U)$-module $\Omega_n(W)$ by

$$\Omega_n(W) = \{w \in W \mid (\mathrm{N}^{n+1}_{\mathsf{L}} U) w = 0\}.$$

We can show that the functors Φ^{L} have the following universal property:

Proposition B.1.4. *Let* M *be a* U*-module and* W_0 *an* $\mathsf{A}_n(U)$*-module. Then there is a natural isomorphism of bifunctors:*

$$\mathrm{Hom}_{\mathsf{A}_n(U)}(W_0, \Omega_n(M)) = \mathrm{Hom}_U(\Phi^{\mathsf{L}}_n(W_0), M).$$

In Sect. 3.1.4 we use Proposition B.1.4 (there given by Proposition 3.1.2) to conclude that Zhu's original induction functor is naturally isomorphic to Φ^{L}.

Proof. We describe the equivalence as follows. For $f\colon W_0 \to \Omega_n(M)$ we define a map $g\colon \Phi^{\mathsf{L}}_n(W_0) \to M$ by $g(u \otimes m) = uf(m)$. Note that if $u \in \mathrm{N}^{n+1}_{\mathsf{L}} U$ then $uf(m) = 0$ as $f(m) \in \Omega_n(M)$. In the other direction, if we are given $g\colon \Phi^{\mathsf{L}}_n(W_0) \to M$, we note that the natural map $W_0 \to \Phi^{\mathsf{L}}_n(W_0)$ defined by $w \mapsto 1 \otimes w$ is injective and by definition of the U-module structure of $\Phi^{\mathsf{L}}_n(W_0)$, has image lying inside $\Omega_n(\Phi^{\mathsf{L}}_n(W_0))$. But as the map g is a U-module map, it follows that $g(W_0) \subset g(\Omega_n(\Phi^{\mathsf{L}}_n(W_0))) \subset \Omega_n(M)$. Consequently we obtain a map $f\colon W_0 \to \Omega_n(M)$ which is easily checked to be an $\mathsf{A}_n(U)$-module map and to give an inverse correspondence to the prior prescription. □

Of course, we can also do a right handed version of this construction for a right $\mathsf{A}_n(U)$ module Z_0 and obtain in this way a right U-module $\Phi^{\mathsf{R}}_n(Z_0)$. We will describe the properties of Φ^{L} and leave the analogue statements about Φ^{R} to the reader.

Lemma B.1.5. *Suppose U is a split-filtered algebra with graded subalgebra U' and with an almost canonical seminorm. Then*

$$\Phi^{\mathsf{L}}(W_0) = \left(U'/\,\mathrm{N}^1_{\mathsf{L}} U'\right) \otimes_{U'_0} W_0 \cong \left(U/\,\mathrm{N}^1_{\mathsf{L}} U\right) \otimes_{U_0} W_0.$$

Proof. This is an immediate consequence of Lemma A.8.1. □

Note that $\mathrm{N}^1_{\mathsf{L}} U$ is a left U module and a right $U_{\leq 0}$ module which is annihilated on the right by $U_{\leq -1}$. In particular, we can also write the above expression as:

$$\left(U/\,\mathrm{N}^1_{\mathsf{L}} U\right) \otimes_{U_0} W_0 \cong \left(U/\,\mathrm{N}^1_{\mathsf{L}} U\right) \otimes_{U_{\leq 0}} W_0,$$

with respect to the truncation quotient map $U_{\leq 0} \to U_0$ with kernel $U_{\leq -1}$. This is because the additional relations in the tensor product on the right are of the form $\alpha\beta \otimes w - \alpha \otimes \overline{\beta} w$ with $\beta \in U_{\leq -1}$. But $\alpha\beta \in UU_{\leq -1} \in \mathrm{N}^1_{\mathsf{L}} U$ represents 0 as does $\overline{\beta}$. Hence these extra relations all vanish.

Remark B.1.6. We see that $\Phi^{\mathsf{L}}(W_0)$ is naturally a graded module, with grading inherited from $U/\,\mathrm{N}^1_{\mathsf{L}} U$:

$$\Phi^{\mathsf{L}}(W_0) = \bigoplus_{p=0}^{\infty} \left(U/\,\mathrm{N}^1_{\mathsf{L}} U\right)_p \otimes_{U_0} W_0 = \bigoplus_{p=0}^{\infty} \left(U_p/\,\mathrm{N}^1_{\mathsf{L}} U_p\right) \otimes_{U_0} W_0.$$

Notice here that $U_{-m} \subset \mathrm{N}^m_{\mathsf{L}} U$ and so $\left(U/\,\mathrm{N}^1_{\mathsf{L}} U\right)_p = 0$ for $p < 0$.

Lemma B.1.7. *The action of U_0 on $\Phi^{\mathsf{L}}(W_0)$ via its left module structure induces an $\mathsf{A}_d(U)$ module structure on $\Phi^{\mathsf{L}}(W_0)_{\leq d} = \bigoplus_{p=0}^{d} \Phi^{\mathsf{L}}(W_0)_p$.*

Proof. We have $U_{\leq -d-1}\Phi^{\mathsf{L}}(W_0)_{\leq d} = 0$ from degree considerations. It follows that

$$({}^{c}\mathrm{N}^{d+1} U_0)\Phi^{\mathsf{L}}(W_0)_{\leq d} = 0.$$

But by Lemma A.6.10 ${}^{c}\mathrm{N}^{d+1} U_0$ is dense in $\mathrm{N}^{d+1} U_0$ and by Lemma A.6.5, the multiplication action of U_0 on $U_{\leq d}$ is continuous, and hence so is the multiplication of U_0 on $U_{\leq d}/\,\mathrm{N}^1_{\mathsf{L}} U_{\leq d}$ and hence of U_0 on $\Phi^{\mathsf{L}}(W_0)$. But as $U/\,\mathrm{N}^1_{\mathsf{L}} U$ has a discrete topology, so does $\Phi^{\mathsf{L}}(W_0)$. Since a dense subset of $\mathrm{N}^{d+1} U_0$ acts as zero, it follows that it acts as zero, making the action of the algebra $\mathsf{A}_d(U)$ well-defined. □

B.2. Generalized Mode Transition Algebras. Lemma B.2.1 is the main technical tool used to define algebraic structures and their actions on generalized Verma modules:

Lemma B.2.1. *Suppose U is a graded algebra with an almost canonical seminorm. Then we have a natural isomorphism*

$$\left(U/\mathrm{N}^1_{\mathsf{R}}U\right) \otimes_U \left(U/\mathrm{N}^1_{\mathsf{L}}U\right) \to \mathsf{A}_0(U), \qquad \overline{\alpha} \otimes \overline{\beta} \mapsto \alpha \circledast \beta$$

where for $\alpha, \beta \in U$ homogeneous, we define $\alpha \circledast \beta$ as follows:

$$\alpha \circledast \beta = \begin{cases} 0 & \textit{if } \deg(\alpha) + \deg(\beta) \neq 0 \\ [\alpha\beta] & \textit{if } \deg(\alpha) + \deg(\beta) = 0 \end{cases}$$

and then extend the definition to general products by linearity.

Proof. As our seminorm is almost canonical, the map $U_0 \to U/\left(\mathrm{N}^1_{\mathsf{L}}U + \mathrm{N}^1_{\mathsf{R}}U\right)$ factors through $U \to U/\left(UU_{\leq -1} + U_{\geq 1}U\right)$. But for this map, we see that both $U_{\leq -1}$ and $U_{\geq 1}$ are in the kernel, which implies that the restriction to U_0 is surjective. The kernel of this map $U_0 \to U/\left(\mathrm{N}^1_{\mathsf{L}}U + \mathrm{N}^1_{\mathsf{R}}U\right)$ consists of $\mathrm{N}^1_{\mathsf{L}}U_0 \cap \mathrm{N}^1_{\mathsf{R}}U_0 = \mathrm{N}^1U_0$ (see Remark A.9.4). □

As an application of the above result, we obtain the following.

Corollary B.2.2. *Let W_0 be a left $\mathsf{A}_0(U)$-module and Z_0 be a right $\mathsf{A}_0(U)$-module. Then the map defined in Lemma B.2.1 induces an isomorphism*

$$\Phi^{\mathsf{R}}(Z_0) \otimes_U \Phi^{\mathsf{L}}(W_0) \to Z_0 \otimes_{\mathsf{A}_0(U)} W_0.$$

Definition B.2.3. For a graded algebra with almost canonical seminorm U, and an $\mathsf{A}_0(U)$-bimodule B, we define a bigraded group:

$$\begin{aligned}\Phi(B) = \Phi^{\mathsf{R}}(\Phi^{\mathsf{L}}(B)) = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(B)) &= \left(U/\mathrm{N}^1_{\mathsf{L}}U\right) \otimes_{U_0} B \otimes_{U_0} \left(U/\mathrm{N}^1_{\mathsf{R}}U\right) \\ &= \bigoplus_{d_1 \geq 0} \bigoplus_{d_2 \leq 0} \left(U/\mathrm{N}^1_{\mathsf{L}}U\right)_{d_1} \otimes_{U_0} B \otimes_{U_0} \left(U/\mathrm{N}^1_{\mathsf{R}}U\right)_{d_2}.\end{aligned}$$

We now introduce the space $\Phi(B)$ and the operation $\star$ arising from $\circledast$ which, as we show below, defines an algebra structure on $\Phi(B)$ whenever B is an associative ring admitting a homomorphism $f\colon \mathsf{A}_0(U) \to B$.

Definition B.2.4. Let B be an associative ring admitting a homomorphism $f\colon \mathsf{A}_0(U) \to B$ and let W_0 be a left B-module. Then we can define a map $\star\colon \Phi(B) \times \Phi^{\mathsf{L}}(W_0) \to \Phi^{\mathsf{L}}(W_0)$ as follows. For $x = \alpha \otimes a \otimes \alpha' \in \Phi(B)$ and $\beta \otimes w \in \Phi^{\mathsf{L}}(W_0)$ we set

$$x \star (\beta \otimes w) = \alpha \otimes a f(\alpha' \circledast \beta) w.$$

Proposition B.2.5. *Via the identification $\Phi(B) = \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(B))$, the map defined in Definition B.2.4 defines an associative algebra structure on $\Phi(B)$ such that $\Phi^{\mathsf{L}}(W_0)$ becomes a left $\Phi(B)$-module. Moreover, $\gamma \cdot (x \star y) = (\gamma \cdot x) \star y$ for every $x \in \Phi(B)$, $y \in \Phi^{\mathsf{L}}(W_0)$, and $\gamma \in U$. Analogously, $(x \star y) \cdot \gamma = x \star (y \cdot \gamma)$ for every $x, y \in \Phi(B)$ and $\gamma \in U$. Finally, with respect to the bigrading of Definition B.2.3, we have*

$$\begin{aligned}&\Phi(U)_{d_1,d_2} \star \Phi(U)_{d_3,d_4} \subseteq \Phi(U)_{d_1,d_4} \qquad \textit{and} \\ &\Phi(U)_{d_1,d_2} \star \Phi(U)_{d_3,d_4} = 0 \qquad \textit{whenever } d_2 + d_3 \neq 0.\end{aligned}$$

Proof. We check first that this satisfies the standard associativity relationship for a module action. Let $\alpha \otimes a \otimes \alpha', \beta \otimes b \otimes \beta' \in \Phi(B)$ and $\gamma \otimes c \in \Phi^{\mathsf{L}}(W_0)$, then

$$\begin{aligned}(\alpha \otimes a \otimes \alpha') \star \big((\beta \otimes b \otimes \beta') \star (\gamma \otimes c)\big) &= (\alpha \otimes a \otimes \alpha') \star (\beta \otimes bf(\beta' \circledast \gamma)c) \\ &= (\alpha \otimes af(\alpha' \circledast \beta)bf(\beta' \circledast \gamma)c) \\ &= (\alpha \otimes af(\alpha' \circledast \beta)b \otimes \beta') \star (\gamma \otimes c) \\ &= \big((\alpha \otimes a \otimes \alpha') \star (\beta \otimes b \otimes \beta')\big) \star (\gamma \otimes c),\end{aligned}$$

as desired. The associativity of the algebra structure now follows, taking $W_0 = \Phi^{\mathsf{R}}(B)$.

We now check the compatibility with the U-module structure on the left. Set $x = \alpha \otimes a \otimes \alpha'$ and $y = \beta \otimes b$. Then

$$\begin{aligned}\gamma \cdot \big((\alpha \otimes a \otimes \alpha') \star (\beta \otimes b)\big) &= \gamma \cdot (\alpha \otimes af(\alpha' \circledast \beta)b) \\ &= (\gamma\alpha) \otimes af(\alpha' \circledast \beta)b = ((\gamma\alpha) \otimes a \otimes \alpha') \star (\beta \otimes b).\end{aligned}$$

The other-handed version of the above argument gives us the compatibility with the U-module structure on the right when $W_0 = \Phi^{\mathsf{R}}(B)$.

The last assertion follows from the product $\circledast$ as described in Lemma B.2.1. □

Definition B.2.6. For a graded algebra with almost canonical seminorm U, we define $\mathfrak{A}(U) = \Phi(\mathsf{A}_0(U))$ to be the *(generalized) mode transition algebra*, and we write $\mathfrak{A}(U)_d$ for the d-th mode transition subalgebra $\mathfrak{A}(U)_{d,-d}$.

In this case, it turns out that the action of $\mathfrak{A}(U)$ extends to much more general modules than simply the Verma-type modules, as we now observe.

Lemma B.2.7. *Suppose U is a graded algebra with a good seminorm, and let W be an $\mathbb{N}$-graded U-module endowed with the discrete topology, such that the action of $U \times W \to W$ is continuous. Then for all $d \in \mathbb{N}$, $\mathsf{N}^{m+1}_{\mathsf{R}} U_{m-d} W_d = 0 = \mathsf{N}^{d+1}_{\mathsf{L}} U_{m-d} W_d$.*

Proof. As a good seminorm is almost canonical, we may iteratively apply Lemma A.6.9, to see that ${}^{c}\mathsf{N}^{m+1}_{\mathsf{R}} U_{m-d}$ is dense in $\mathsf{N}^{m+1}_{\mathsf{R}} U_{m-d}$. Consequently, if $\alpha \in \mathsf{N}^{m+1}_{\mathsf{R}} U_{m-d}$ and $w \in W_d$, then by continuity, we can find $\alpha' \in {}^{c}\mathsf{N}^{m+1}_{\mathsf{R}} U_{m-d}$ such that $\alpha' w = \alpha w$. But therefore $\alpha' w \in UU_{-d-1} w = 0$. As the seminorm is good, we find $\mathsf{N}^{d+1}_{\mathsf{L}} U_{m-d} = \mathsf{N}^{d+1}_{\mathsf{R}} U_{m-d}$, giving the final claim. □

Lemma B.2.8. *Suppose U is a graded algebra with a good seminorm, and let W be an $\mathbb{N}$-graded U-module endowed with the discrete topology, such that the action of $U \times W \to W$ is continuous. Then the left action of U on W induces well-defined maps for every p:*

$$\begin{aligned}(U_p / \mathsf{N}^{p+d+1}_{\mathsf{R}} U_p) \times W_d &\to W_{p+d}, \\ ([u], w) &\mapsto uw,\end{aligned}$$

which we will simply write as $[u]w$.

Proof. This follows from Lemma B.2.7, which shows $(\mathsf{N}^{p+d+1}_{\mathsf{R}} U_p) W_d = 0$. □

Proposition B.2.9. *Let U be a graded algebra with an almost canonical seminorm, and let W be an $\mathbb{N}$-graded U-module endowed with the discrete topology, such that the action of $U \times W \to W$ is continuous. Then we obtain an action of $\mathfrak{A}(U)$ on W such that $\mathfrak{A}(U)_{d_1,-d_2} W_{d_2} \subset W_{d_1}$ and $\mathfrak{A}(U)_{d_1,-d_2} W_{d_3} = 0$ when $d_2 \neq d_3$ which, on simple tensors, is given by the products defined in Lemma B.2.8:*

$$(\alpha \otimes a \otimes \alpha') \star w = \begin{cases} \alpha(a(\alpha' w)) & \text{if } d_2 = d_3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Explicitly, if we write $\alpha = [u], a = [x], \alpha' = [v]$ for $u \in U_{d_1}, v \in U_{-d_2}, x \in U_0$, we have by definition (Lemma B.2.8), $\alpha(a(\alpha' w)) = u(x(v(w)))$. It follows from Lemma B.2.8 that this is a well-defined map. Further, associativity of these products follows from the associativity of the action of U on W. □

B.3. Relationship with Higher Generalized Zhu Algebras. Throughout this section, let us fix a graded algebra U complete with respect to an almost canonical seminorm. We will write A_n in place of $\mathsf{A}_n(U)$ and $\mathfrak{A}_n$ in place of $\mathfrak{A}(U)_n$.

The action of U_0 on $\mathfrak{A}_n$ is continuous where $\mathfrak{A}_n$ is defined as a discrete module.

Lemma B.3.1. *For each $d \geq 0$, there is an exact sequence*

$$\mathfrak{A}_d \xrightarrow{\mu_d} \mathsf{A}_d \xrightarrow{\pi_d} \mathsf{A}_{d-1} \longrightarrow 0, \tag{22}$$

where $\mu_d(\overline{\alpha} \otimes [u]_0 \otimes \overline{\beta}) = [\alpha u \beta]_d$, for all $\alpha \in U_d$ (respectively $\beta \in U_{-d}$) and where $\overline{\alpha}$ (respectively $\overline{\beta}$) denotes its class in $U/\mathrm{N}^1_{\mathsf{L}} U$ (respectively in $U/\mathrm{N}^1_{\mathsf{R}} U$).

Proof. We first check that the map μ_d is well-defined. Notice μ_d is independent on the lifts of $\overline{\alpha}$ and $\overline{\beta}$ to U, since $\mathrm{N}^1_{\mathsf{L}} U_{-d} \cdot U_0 \cdot U_d \subseteq \mathrm{N}^1 U_0$ and similarly $U_d \cdot U_0 \cdot \mathrm{N}^1_{\mathsf{R}} U_{-d} \subseteq \mathrm{N}^1 U_0$. Analogously, since $U_d \cdot \mathrm{N}^1 U_0 \cdot U_{-d} \subseteq \mathrm{N}^d U_0$, the map μ_d is independent of the lift of $[u]_0$ to $u \in U_0$. Finally, we need to show that it respects the tensor products over U_0. For this we need to check that $\mu_d(\overline{\alpha v} \otimes [u]_0 \otimes \overline{\beta}) = \mu_d(\overline{\alpha} \otimes [vu]_0 \otimes \overline{\beta})$ for every $v \in U_0$. But by definition both are the class of the element $\alpha v u \beta$ in A_d.

We have identifications $\mathsf{A}_d = U_0/\mathrm{N}^{d+1} U_0$ and $\mathsf{A}_{d-1} = U_0/\mathrm{N}^d U_0$. Consequently the kernel of the canonical projection π_d can be written as $\mathrm{N}^d U_0/\mathrm{N}^{d+1} U_0$. It follows from the definition of μ_d and $\mathfrak{A}_d$ that the image of the μ_d consists exactly of sums of elements of the form $[\alpha\beta]_d$ with $\deg \alpha = d$ and $\deg \beta = -d$. Hence the image of μ_d consists of the image of ${}^c\mathrm{N}^d U_0$ in A_d. Since we have an almost canonical filtration, we have by Lemma A.6.9,

$$\mathrm{N}^d U_0 = {}^c\mathrm{N}^d U_0 + \mathrm{N}^{d+1} U_0,$$

which shows that μ_d is therefore surjective onto $\mathrm{N}^d U_0/\mathrm{N}^{d+1} U_0 = \ker \pi_d$, showing right exactness. □

The following result is immediate from the definitions and from the associativity of the actions.

Lemma B.3.2. *Let W_0 be an A_0-module. Then the action of $\mathfrak{A}_d$ on $\Phi^{\mathsf{L}}(W_0)_d$ factors through the action of A_d described in Lemma B.1.7 via the map μ_d.*

We now are ready to state the principal result of this section.

Theorem B.3.3. *If $\mathfrak{A}_d$ admits an identity element, then the map μ_d in (22) is injective and the sequence splits, giving a ring product $\mathsf{A}_d \cong \mathfrak{A}_d \times \mathsf{A}_{d-1}$.*

Proof. We first check that the map μ_d is injective. Suppose we are given an element $\mathfrak{a} \in \mathfrak{A}_d$ which is in the kernel of this map. By Lemma B.3.2, the action of $\mathfrak{a}$ on $\Phi^{\mathsf{L}}(M)_d$ for any M factors through the action of A_d via μ_d, so it should be 0. If we consider the case of $M = \Phi^{\mathsf{R}}(\mathsf{A}_0)$, this says that, in particular, the action of $\mathfrak{a}$ on $\mathfrak{A}_d \subset \Phi^{\mathsf{L}}(\Phi^{\mathsf{R}}(\mathsf{A}_0))_d$ is 0. This action is identified with the algebra product via Definition B.2.4. It follows that since $\mathfrak{A}_d$ has an identity element $\mathscr{I}_d$, we have $\mathfrak{a} = \mathfrak{a} \star \mathscr{I}_d = 0$ as claimed.

Since μ_d is injective we will omit it in the remainder of the proof and see $\mathfrak{A}_d$ as naturally sitting inside A_d. Denote the unity in the higher level Zhu algebras A_d by 1, and write $e = \mathscr{I}_d$. Let $f = 1 - e$ so that e and f are orthogonal idempotents. Note that e generates the 2-sided ideal $\mathfrak{A}_d \triangleleft \mathsf{A}_d$. Furthermore, since e is the unity of $\mathfrak{A}_d$, for every $a \in \mathsf{A}_d$, we have $ae = eae = ea$ and so $e \in Z(\mathsf{A}_d)$. Consequently $f = 1 - e \in Z(\mathsf{A}_d)$ as well. It follows that f and e are orthogonal central idempotents, and therefore $\mathsf{A}_d = \mathsf{A}_d e \times \mathsf{A}_d f$ as rings. But $\mathsf{A}_d e = \mathfrak{A}_d$ and $\mathsf{A}_d f \cong \mathsf{A}_d/\mathfrak{A}_d \cong \mathsf{A}_{d-1}$ completing our proof. □

References

[AB23a] Addabbo, D., Barron, K.: The level two Zhu algebra for the Heisenberg vertex operator algebra. Comm. Algebra **51**(8), 3405–3463 (2023)

[AB23b] Addabbo, D., Barron, K.: On generators and relations for higher level Zhu algebras and applications. J. Algebra **623**, 496–540 (2023)

[AM07] Adamovic, D., Milas, A.: Logarithmic intertwining operators and $\mathcal{W}(2, 2p\text{-}1)$ algebras. J. Math. Phys. **48**(7), 073503, 20 (2007)

[AM10] Adamović, D., Milas, A.: On W-algebras associated to $(2, p)$ minimal models and their representations. Int. Math. Res. Not. IMRN **20**, 3896–3934 (2010)

[AM11] Adamović, D., Milas, A.: The structure of Zhu's algebras for certain $\mathcal{W}$-algebras. Adv. Math. **227**(6), 2425–2456 (2011)

[BCHM10] Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Amer. Math. Soc. **23**(2), 405–468 (2010)

[BFM91] Beilinson, A., Feigin, B., Mazur, B.: Introduction to algebraic field theory on curves. Unpublished (1991)

[Buh02] Buhl, G.: Spanning set for VOA modules. J. Algebra **254**(1), 125–151 (2002)

[BVWY19a] Barron, K., Werf, N.V., Yang, J.: Higher level Zhu algebras and modules for vertex operator algebras. J. Pure Appl. Algebra **223**(8), 3295–3317 (2019)

[BVWY19b] Barron, K., Vander Werf, N., Yang, J.: The level one Zhu algebra for the Heisenberg vertex operator algebra. In: Affine, vertex and W-algebras, volume 37 of Springer INdAM Ser., pp. 37–64. Springer, Cham, [2019] (2019)

[BVWY20] Barron, K., Vander Werf, N., Yang, J.: The level one Zhu algebra for the Virasoro vertex operator algebra. In Vertex operator algebras, number theory and related topics, volume 753 of Contemp. Math., pages 17–43. Amer. Math. Soc., [Providence], RI, [2020] (2020)

[CKL20] Creutzig, T., Kanade, S., Linshaw, A.R.: Simple current extensions beyond semi-simplicity. Commun. Contemp. Math. **22**(1), 1950001, 49 (2020)

[Cod19] Codogni, G.: Vertex algebras and Teichmüller modular forms. arXiv:1901.03079, pp. 1–31 (2019)

[DG23] Damiolini, C., Gibney, A.: On global generation of vector bundles on the moduli space of curves from representations of vertex operator algebras. Algebr. Geom. **10**(3), 298–326 (2023)

[DGK22] Damiolini, C., Gibney, A., Krashen, D.: Factorization presentations. arXiv:2207.05110, (2022)

[DGK24] Damiolini, C., Gibney, A., Krashen, D.: Morita equivalences for Zhu's associative algebra and mode transition algebras. arXiv:2403.11855, (2024)

[DGT21] Damiolini, C., Gibney, A., Tarasca, N.: Conformal blocks from vertex algebras and their connections on $\overline{\mathcal{M}}_{g,n}$. Geom. & Topol. **25**(5), 2235–2286 (2021)

[DGT22] Damiolini, C., Gibney, A., Tarasca, N.: Vertex algebras of CohFT-type. In Facets of algebraic geometry. Vol. I, volume 472 of London Math. Soc. Lecture Note Ser., pages 164–189. Cambridge Univ. Press, Cambridge (2022)

[DGT24] Damiolini, C., Gibney, A., Tarasca, N.: On factorization and vector bundles of conformal blocks from vertex algebras. Ann. Sci. Éc. Norm. Supér. (4) **57**(1), 241–292 (2024)
[DJ08] Dong, C., Jiang, C.: Bimodules associated to vertex operator algebras. Math. Z. **259**(4), 799–826 (2008)
[DL93] Dong, C., Lepowsky, J.: Generalized vertex algebras and relative vertex operators. Prog in Maths, vol. 112. Birkhäuser Boston, MA (1993)
[DLM97] Dong, C., Li, H., Mason, G.: Regularity of rational vertex operator algebras. Adv. Math. **132**(1), 148–166 (1997)
[DLM98] Dong, C., Li, H., Mason, G.: Vertex operator algebras and associative algebras. J. Algebra **206**(1), 67–96 (1998)
[EH87] Eisenbud, D., Harris, J.: The Kodaira dimension of the moduli space of curves of genus ≥ 23. Invent. Math. **90**(2), 359–387 (1987)
[ELSV01] Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A.: Hurwitz numbers and intersections on moduli spaces of curves. Invent. Math. **146**(2), 297–327 (2001)
[Far09] Farkas, G.: Koszul divisors on moduli spaces of curves. Amer. J. Math. **131**(3), 819–867 (2009)
[FBZ04] Frenkel, E., Ben-Zvi, D.: Vertex algebras and algebraic curves, volume 88 of Mathematical Surveys and Monographs. Am Mathem Soc, Providence, RI, second edition, (2004)
[FGST06] Feigin, B.L., Gainutdinov, A.M., Semikhatov, A.M., Tipunin, I.Y.: Logarithmic extensions of minimal models: characters and modular transformations. Nuclear Phys. B **757**(3), 303–343 (2006)
[FHL93] Frenkel, I., Huang, Y.Z., Lepowsky, J.: On axiomatic approaches to voas and modules. Mem. Amer. Math. Soc. **104**(494), viii+64 (1993)
[Fre07] Frenkel, E.: Langlands correspondence for loop groups. Cambridge Studies in Advanced Mathematics, vol. 103. Cambridge University Press, Cambridge (2007)
[FZ92] Frenkel, I.B., Zhu, Y.: Voas associated to representations of affine and Virasoro algebras. Duke Math. J. **66**(1), 123–168 (1992)
[GN03] Gaberdiel, M.R., Neitzke, A.: Rationality, quasirationality and finite W-algebras. Comm. Math. Phys. **238**(1–2), 305–331 (2003)
[Gro67] Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math. **32**, 361 (1967)
[Har77] Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg (1977)
[He17] He, X.: Higher level Zhu algebras are subquotients of universal enveloping algebras. J. Algebra **491**, 265–279 (2017)
[HLZ06] Huang, Y.-Z., Lepowsky, J., Zhang, L.: A logarithmic generalization of tensor product theory for modules for a vertex operator algebra. Internat. J. Math. **17**(8), 975–1012 (2006)
[HLZ14] Huang, Y. Z., Lepowsky, J., Zhang, L.: Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: introduction and strongly graded algebras and their generalized modules. In: Conformal field theories and tensor categories, Math. Lect. Peking Univ., pages 169–248. Springer, Heidelberg (2014)
[HM82] Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**(1), 23–88 (1982). (**With an appendix by William Fulton**)
[Hua05] Huang, Y.-Z.: Differential equations, duality and modular invariance. Commun. Contemp. Math. **7**(5), 649–706 (2005)
[Hua20] Huang, Y.Z.: Associative algebras and the representation theory of grading-restricted vertex algebras. arXiv:2009.00262v3 (2020)
[Hua21] Huang, Y.Z.: Associative algebras and intertwining operators. arXiv:2111.06943v2 (2021)
[Hua23] Huang, Y.Z.: Modular invariance of (logarithmic) intertwining operators. arXiv:2305.15152v1 (2023)
[Li94] Li, H.: Representation theory and tensor product theory for vertex operator algebras. ProQuest LLC, Ann Arbor, MI (1994). Thesis (Ph.D.)–Rutgers - New Brunswick
[Li96] Li, H.-S.: Local systems of vertex operators, vertex superalgebras and modules. J. Pure Appl. Algebra **109**(2), 143–195 (1996)
[LL04] Lepowsky, J., Li, H.: Intro to vertex algebras and their representations. Progress in Maths, vol. 227. Birkhäuser, Boston, MA (2004)
[Mil08] Milas, A.: Logarithmic intertwining operators and vertex operators. Comm. Math. Phys. **277**(2), 497–529 (2008)
[Miy04] Miyamoto, M.: Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. Duke Math. J. **122**(1), 51–91 (2004)
[MNT10] Matsuo, A., Nagatomo, K., Tsuchiya, A.: Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. In Moonshine: the first quarter century and beyond, volume 372 of LMS. Lecture Note Ser., pp. 282–329. Cambridge Univ. Press, Cambridge (2010)

[MOP+17] Marian, A., Oprea, D., Pandharipande, R., Pixton, A., Zvonkine, D.: The Chern character of the Verlinde bundle over $\overline{\mathcal{M}}_{g,n}$. J. Reine Angew. Math. **732**, 147–163 (2017)

[Mum83] Mumford, D.: Towards an enumerative geometry of the moduli space of curves. In: Arithmetic and geometry, Vol. II, volume 36 of Progr. Math., pages 271–328. Birkhäuser Boston, Boston, MA (1983)

[NT05] Nagatomo, K., Tsuchiya, A.: Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line. Duke Math. J. **128**(3), 393–471 (2005)

[NT11] Nagatomo, K., Tsuchiya, A.: The triplet vertex operator algebra $W(p)$ and the restricted quantum group $\overline{U}_q(sl_2)$ at $q = e^{\frac{\pi i}{p}}$. In Exploring new structures and natural constructions in mathematical physics, volume 61 of Adv. Stud. Pure Math., pages 1–49. Math. Soc. Japan (2011)

[Pri00] Pries, R.J.: Construction of covers with formal and rigid geometry. In Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998), volume 187 of Progr. Math., pages 157–167. Birkhäuser, Basel (2000)

[Sta23] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2023

[TUY89] Tsuchiya, A., Ueno, K., Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries. In Integrable systems in quantum field theory and statistical mechanics, volume 19 of Adv. Stud. Pure Math., pages 459–566. Ac Pr, Boston, MA (1989)

[TW13] Tsuchiya, A., Wood, S.: The tensor structure on the representation category of the $\mathcal{W}_p$ triplet algebra. J. Phys. A **46**(44), 445203 (2013)

[vE11] van Ekeren, J.: Higher level twisted zhu algebras. J. Math. Phys. (2011). https://doi.org/10.1063/1.3589214

[vEH19] van Ekeren, J., Heluani, R.: A short construction of the zhu algebra. J. Algebra **528**, 85–95 (2019)

[Wan93] Wang, W.: Rationality of Virasoro vertex operator algebras. Internat. Math. Res. Notices **7**, 197–211 (1993)

[Wei94] Weibel, C.A.: An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, vol. 38. Cambridge University Press, Cambridge (1994)

[Zhu96] Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. **9**(1), 237–302 (1996)

Communicated by C. Schweigert