Line bdles on the moduli of parahoric bundles

with J. Hong

$\mathrm{Bun}_{\mathcal{G}}$ = $\boxed{\mathcal{G}}$ - bundles on X = smooth proj curve / $\mathbb{C}$

Parahoric Bruhat - Tits group (group scheme over X)

[1] CONSTANT CASE

$\mathcal{G} = G \times X$ $G/\mathbb{C}$ simple & simply conn

from now on G will always denote such a group

eg: SL_r v.b. of rk r + trivial determinant

[2] PARABOLIC CASE

$p_1 \cdots p_n \in X$, $\mathring{X} = X \setminus \{p_1 \cdots p_n\}$

$\mathcal{G}|_{\mathring{X}} = G \times \mathring{X}$ constant

$\mathcal{G}(\mathbb{D}_{p_i}) = \underset{\text{local coordinate}}{\mathcal{G}(\mathbb{C}[[t_i]])} = ev^{-1}(B)$

$ev: G(\mathbb{C}[[t_i]]) \to G(\mathbb{C})$
$t_i \longmapsto 0$

Thm[B.T] (Bruhat - Tits) $\exists!$ smooth affine alg. group $\mathcal{G}$ over X that satisfies $\uparrow$

SL_r 

$\mathrm{Bun}_{\mathcal{G}} = \mathrm{ParBun}_G = \{\mathcal{E} \in G\text{-bdle} + s_i \in (\mathcal{E}/B)(p_i)\}$

[3] TWISTED CASE

$\pi: \tilde{X} \longrightarrow X$ Γ-cover might be ramified

$\Gamma \longrightarrow \mathrm{Aut}(G)$

$$\mathcal{G} = (\pi_*(\tilde{X} \times G))^{\Gamma}$$

note that $\mathcal{G}|_p = G$ if p not ramified, but at ramification points $\mathcal{G}|_p \neq G$ in general!

Definition $\mathcal{G}$ over X is Parahoric BT (PBT) if

- $\mathcal{G}$ affine & smooth group scheme $/X$
- generically simply conn & simple — fiber has root datum which gives G
- fibers are connected
- $\mathcal{R}$ = "bad points of $\mathcal{G}$:= $\mathcal{G}|_{p_i}$ not reductive" — $\mathrm{Spec}(\mathbb{C}((t)))$

$\forall p \in \mathcal{R}$ $\mathcal{G}(\mathbb{D}_p)$ is $\boxed{\text{Parahoric sub}}$ of $\mathcal{G}(\mathbb{D}_p^\times)$

$\mathrm{Spec}(\mathbb{C}[[t]])$

this means that it has the shape given as follows:

$$\mathcal{G}(\mathbb{D}_p) \cong G(\mathbb{C}[[t^{1/m}]])^\sigma \to G(\mathbb{C}((t^{1/m})))^\sigma \xleftarrow[\cong]{Ad_{t^{\lambda/m}}} G(\mathbb{C}((t^{1/r})))^\tau = \mathcal{G}(\mathbb{D}_p^\times)$$

τ diagr. aut of
$\tau^r = id$

$\sigma = Ad_h \circ \tau$ $h \in T^\tau_{ad}$
$\sigma^m = 1$ $h = \zeta_m^\lambda$ $\lambda \in$ cochar

NOTE the definition of parahoric subgroup given by Bruhat & Tits is different & uses buildings. However we will use this equivalent incarnation of the concept

QUESTIONS [Pappas-Rapoport ~2008]

(I) can we describe $\mathrm{Pic}(\mathrm{Bun}_\mathcal{G})$ & (II) sections using representation theory?

smooth alg stack loc finite type

main tool: $\boxed{\text{TWISTED CONFORMAL BLOCKS}}$

UNIFORMIZATION Thm:

Assume: $\mathcal{R} \neq \emptyset$ [set of bad points] (or if not $\mathcal{R} = x \in X$)

$\mathring{X} = X \setminus \mathcal{R}$

Thm: [Heinloth, Drinfeld-Simpson, Beauville-Laszlo]

$$\mathrm{Bun}_\mathcal{G} = \mathcal{G}(\mathring{X}) \backslash \left[\prod_{p\in\mathcal{R}} \mathcal{G}(\mathbb{D}_p^\times) \big/ \prod \mathcal{G}(\mathbb{D}_p) \right] = LG/L^+G$$

$$\left[= \prod_{p\in\mathcal{R}} \mathrm{Gr}_{\mathcal{G},p} \right]$$

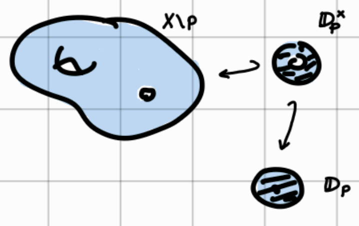
local object $\mathrm{Gr}_\mathcal{G}$ $/\!/$

"every $\mathcal{G}$ bdle is trivializable on $\mathring{X}$ & $\coprod \mathbb{D}_p^\times$"

Ex: SL_r-bdle: vb + trivial det it is trivial on $X\setminus p$ & on $\mathbb{D}_p$

on Dedekind domains proj + det free $\Rightarrow$ free

$\Rightarrow$ only gluing $SL_r(\mathbb{D}_p^\times)$



i.e. $\mathrm{Gr}_{\mathcal{G}} \xrightarrow{q} \mathrm{Bun}_{\mathcal{G}}$ it is a $\mathcal{G}(\mathring{X})$-torsor (fpqc top)

$$\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) = \underbrace{\boxed{a}\ \text{line bdles on } \mathrm{Gr}_{\mathcal{G}}}_{\text{we'll see how these, and their global sections, are described using rep thy of (twisted) affine Lie algebras}} + \underbrace{\boxed{b}\ \mathcal{G}(\mathring{X})\text{-linearization}}_{\text{this is where ALL the geometry appears, but since } \mathring{X}=X\setminus\mathcal{R}\text{, we got rid of the bad locus} \rightsquigarrow \text{more tractable and in fact we get}},$$

$$q^*: \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) \longrightarrow \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})$$

PROP [DH] q^* is injective, i.e. at most a line bdle on $\mathrm{Gr}_{\mathcal{G}}$ admits one unique linearization.

proof: enough to show $L_{\mathring{X}}\mathcal{G}$ has no characters

lemma on $\mathring{X} = X\setminus\mathcal{R}$
$\mathcal{G}|_{\mathring{X}} = \pi_*(G\times\mathring{C})^{\Gamma}$ for $\mathring{C} \xrightarrow{\Gamma} \mathring{X}$ connected étale & $\Gamma \subseteq \mathrm{Diag}(G)$

$$\begin{array}{ccc}
\mathcal{E} = \mathrm{Iso}(\mathcal{G}, G\times\mathring{X}) & \supseteq & \mathcal{E}^{\circ} \\
\downarrow{\scriptstyle \mathrm{Aut}\,G} & & \downarrow \\
\mathring{X} & \xleftarrow{\ \Gamma\ } & \mathring{C} = \mathcal{E}^{\circ}/G_{ad} = (\mathcal{E}/G_{ad})^{\circ}
\end{array}$$

$0 \to G_{ad} \hookrightarrow \mathrm{Aut}\,G \to \mathrm{Diag} \to 0$
Γ = stabilizer of $(\mathcal{E}/G_{ad})^{\circ}$

[Hong-Kumar] $\pi_*(G\times\mathring{C})^{\Gamma}$ is an ind-integral group scheme (*)

$\Rightarrow$ only trivial characters

if $\mathcal{G}(\mathring{X}) \xrightarrow{\chi} \mathbb{G}_m$, then $d\chi : \mathrm{Lie}(L_{\mathring{X}}\mathcal{G}) \to \mathbb{C}$ is induced
but $[\mathrm{Lie}, \mathrm{Lie}] = \mathrm{Lie}$ & so $d\chi = 0 \Rightarrow \chi$ constant by integr.
$\mathcal{G}$ gen simple

□

$$H^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}) = H^0(\mathrm{Gr}_{\mathcal{G}}, q^*\mathcal{L})^{\mathcal{G}(\mathring{X})} \overset{*}{=} H^0(\mathrm{Gr}_{\mathcal{G}}, q^*\mathcal{L})^{\mathrm{Lie}\,\mathcal{G}(\mathring{X})}$$

since $\mathcal{G}(\mathring{X})$ ind-integral,
taking invariants under $\mathcal{G}(\mathring{X})$ = taking invariants under its Lie Algebra

$\boxed{a}$ Pappas-Rapoport give the following:

$$\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}) = \prod_{p\in R} \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G},p}) \overset{[PR]}{\cong} \prod_{p\in R} \bigoplus_{i\in Y_p} \mathbb{Z}\Lambda_i$$

Parabolic $\subseteq G$ $\rightsquigarrow$ $\overset{\text{set of}}{\text{vertices}}$ of $I_{\mathfrak{g}}$ = dynkin diagram of Lie G

Parahoric $\subseteq G(\mathbb{C}((t^{1/r})))^{\tau}$ $\rightsquigarrow$ $\overset{\phi\neq\text{ set of}}{\text{vertices}}$ of $\hat{I}_{\tau}$ affine Dynkin diagr of $\hat{G}(\mathbb{C}((t^{1/r})))^{\tau}$

$$\mathcal{G}(\mathbb{D}_p) \longmapsto Y_p$$

[Borel-Weil]: If $\vec{\lambda} \in \mathrm{Pic}_+(\mathrm{Gr}_{\mathcal{G}})$ i.e. $\vec{\lambda} = (\Sigma a_i \Lambda_i)$ $a_i \in \mathbb{N}$

$$(**)\quad H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\lambda}}) = \mathbb{H}_{\vec{\lambda}}^{*}$$ dual of repr of

$$\bigoplus \mathfrak{g}((t^{1/r}))^{\tau} \oplus \mathbb{C} = \mathrm{Lie}(\hat{LG})$$

Q1: now that we know $\mathrm{Pic}\,\mathrm{Gr}_{\mathcal{G}}$, can we say more about descent to $\mathrm{Pic}\,\mathrm{Bun}_{\mathcal{G}}$?

$$c_p : \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G},p}) \xrightarrow{\text{central charge}} \mathbb{Z}$$

$\Lambda_i \longmapsto \check{a}_i$ dual Kac label

($\Lambda_0 \longmapsto 1$ special vertex)

this measures how the center of the twisted affine Lie alg. acts on $\mathbb{H}_{\Lambda_p}$

This gives $\mathrm{Pic}\,\mathrm{Gr}_{\mathcal{G}} = \prod(\bigoplus \mathbb{Z}\Lambda_i) \xrightarrow{c} \prod \mathbb{Z}$

and from here we construct:

$$\begin{tikzcd}
 & & \operatorname{Pic} Gr_G = \prod\left(\bigoplus \mathbb{Z}\Lambda_i\right) \arrow[rr, "e"] & & \prod \mathbb{Z} \\
1 \arrow[r] & \prod \operatorname{Char}(G_p) \arrow[r] \arrow[d, equal] & \operatorname{Pic}^{\Delta} = e^{-1}(\mathbb{Z}) \arrow[u, hook] \arrow[rr] & & c_{\Delta}\mathbb{Z} \arrow[u, "\int \Delta"'] \arrow[r] & 0 & {[PR]} \\
1 \arrow[r] & \prod \operatorname{Char}(G_p) \arrow[r] & \operatorname{Pic} \operatorname{Bun}_G \arrow[u, "q^*", "{[Zhu]}"'] \arrow[rr, "\text{induced}"'] & & c_G \mathbb{Z} \arrow[u] \arrow[r] & 0 & {[H]}
\end{tikzcd}$$

0) 1) 2) 3)

$$c_{\Delta} = \operatorname{lcm}_{p \in R} \left(\operatorname{GCD}_{i \in Y_p} \check{a}_i \right)$$

steps: 0) Define $\operatorname{Pic}^{\Delta} \subseteq \operatorname{Pic} Gr_G$

1) Zhu shows q^* factors through $\operatorname{Pic}^{\Delta}$

2) We have an induced central charge map $\operatorname{Pic} \operatorname{Bun}_G \to \mathbb{Z}$

3) Complete to s.e.s: top by Pappas Rapoport, bottom by Heinloth

c_{Δ} & c_G are positive integers

CONJ [D-Hong]: $\operatorname{Pic}^{\Delta} = q^* \operatorname{Pic}(\operatorname{Bun}_G)$ (or equiv. $c_{\Delta} = c_G$)

Knew already true in [1] & [2], where $c_{\Delta} = c_G = 1$

THM [D-Hong] We know this holds in the following cases:

- If $G = \pi_*(G \times C)^{\Gamma \subseteq \Delta}$ then $\operatorname{Pic} = c_G \mathbb{Z}$ and unless $\Gamma = S_3$ (only if $G = D_4$), we have that
$$c_G = c_{\Delta} = \begin{cases} 2 & \text{if } G = SL_{2r+1} \text{ \& } \pi \text{ ramified} \\ 1 & \text{otherwise} \end{cases}$$

(★) • If $0 \in Y_p$ $\forall p \in R$ (Iwahori case) and the Γ of $\mathring{C} \to \mathring{X}$ is not S_3, then $c_G = c_{\Delta} = 1$

how can we prove such result?

enough to show $\exists\ \underbrace{\mathcal{L} \text{ on } \operatorname{Bun}_G}_{\mathcal{L}_{\vec{\lambda}} \text{ on } Gr_G \,+\, \text{linearization}}$ with $c(\mathcal{L}) = c_{\Delta}$

Before doing this, we turn to Q2: can we compute global sections of line bundles on Bun_G?

Take $\mathcal{L}_{\vec{\Lambda}}$ on Pic^{Δ} with Λ_p dominant (i.e. $a_i \in \mathbb{N}$ at least one $+$)

$$\begin{tikzcd} \mathring{C} \arrow[r, "\mathring{\pi}", "\text{étale}"'] \arrow[d, hook] & \mathring{X} \arrow[d, hook] \\ C \arrow[r, "\pi", "\text{possibly ramified}"'] & X \end{tikzcd}$$

$\mathcal{G}|_{\mathring{X}} = \pi_*(\mathring{C} \times G)^{\Gamma}$ ⊛ lemma above

$\mathcal{H} := \pi_*(C \times G)^{\Gamma}$ $\neq \mathcal{G}$ in general

TWISTED GROUP

$\mathbb{H}_{\vec{\Lambda}}$ is a representation of

$$\hat{\mathfrak{h}} := \bigoplus_{p \in R} \mathrm{Lie}(\mathcal{H})(D_p^{\times}) \oplus \mathbb{C}$$

$$\uparrow \text{ map of Lie Alg}$$

$$\mathrm{Lie}(\mathcal{H})(\mathring{X})$$

$$\mathbb{V}(\mathcal{H}, \vec{\Lambda})_{C \to X} := \frac{\mathbb{H}_{\vec{\Lambda}}}{\mathrm{Lie}(\mathcal{H})(\mathring{X})}$$

this is the space of TWISTED COINVARIANTS assoc. with $\vec{\Lambda}$ and $\mathcal{H}$.

$$\mathbb{V}^{\dagger}(\mathcal{H}, \vec{\Lambda})_{C \to X} := \mathrm{Hom}(\mathbb{V}, \mathbb{C})$$

linear dual is the space of TWISTED CONFORMAL BLOCKS

$\|$ by definition + BW applied to $\mathbb{H}_{\vec{\Lambda}}^*$ (see $(**)$ above)

$$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\Lambda}})^{\mathrm{Lie}(\mathcal{H})(\mathring{X})}$$

$\|$ ⊛

$$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\Lambda}})^{\mathrm{Lie}(\mathcal{G})(\mathring{X})}$$

$\|$ if $\mathcal{L}_{\vec{\Lambda}}$ descends to $\mathrm{Bun}_{\mathcal{G}}$,

$$H^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L})$$

here we use that taking invariants for grp = invariants under Lie Alg if you have integral & only one action of $\mathrm{Lie}(\mathcal{G})(\mathring{X})$ on $H_{\vec{\Lambda}}^*$

UPSHOT: Thm [D-Hong, conj. by P&R]

$$H^0(\mathrm{Bun}_G, \mathcal{L}) = \mathbb{V}^+(\mathcal{H}, \vec{\lambda})$$

RMK: [Hong-Kumar] show $\mathbb{V}^+(\mathcal{H}, \vec{\lambda}) = H^0(\mathrm{ParBun}_{\mathcal{H}}, \mathcal{L})$ where $\vec{\lambda}$ determines $\mathcal{L}$ and parabolic structure

We can use this backwards actually:

<u>Thm</u> [D-Hong] If $\mathbb{V}(\mathcal{H}, \vec{\lambda}) \neq 0$, then $\mathcal{L}_{\vec{\lambda}}$ descends to Bun_G

Idea: nonzero CB allows us to construct a linearization for $\mathcal{L}_{\vec{\lambda}}$. This idea is due to Sorger & was used to show $\mathrm{Pic}\,\mathrm{Gr}_G = \mathrm{Pic}\,\mathrm{Bun}_G$ in cases [1] & [2]