Conformal blocks and moduli spaces

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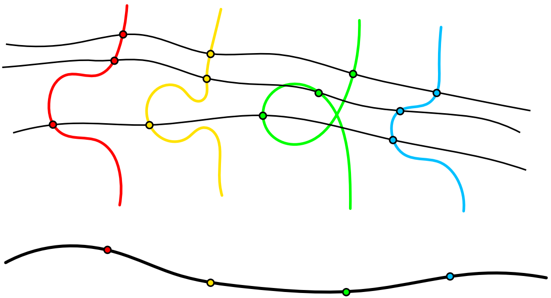

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More generally, we are interested in Bun_G for a simple and simply connected group G: this parametrizes principal G-bundles over C.

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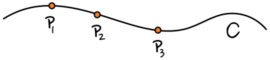

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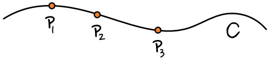

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A simple and simply connected algebraic group G and n irreducible representations $V_1, \dots V_n$ of G of bounded level ℓ.

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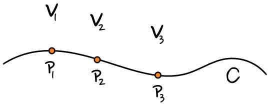

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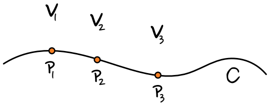

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The dimension of $\mathbb{V}_\ell((C, \underline{P}), (G, \underline{V}))$ is explicitly computed with the Verlinde formula.

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Fix a representation V of G.

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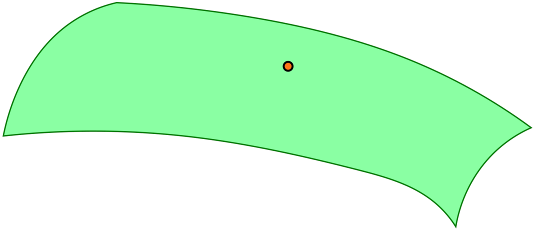

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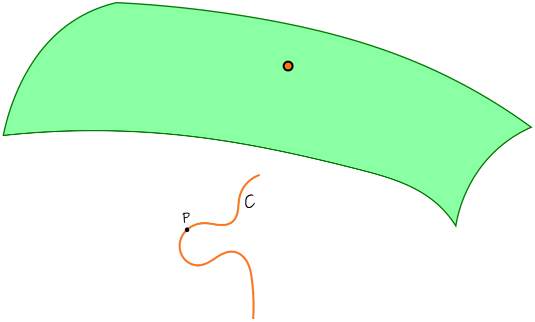

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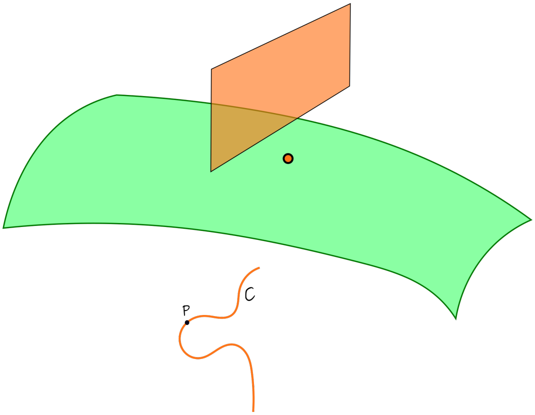

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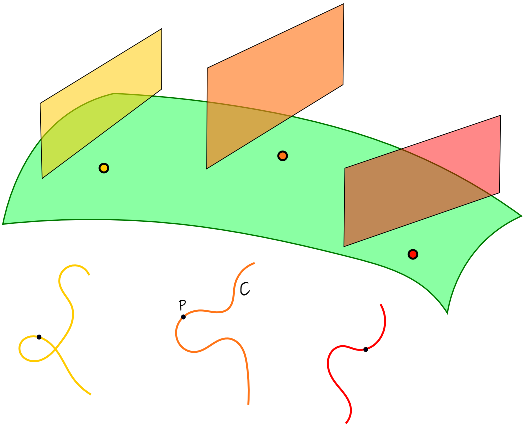

Associating to each pointed curve (C, P) the conformal block $\mathbb{V}_\ell((C, P), (G, V))$ defines the vector bundle

$$\boxed{\mathbb{V}_\ell(V)}$$

on $\overline{M}_{g,1}$.

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Example. Let C be a curve of genus g, then the Verlinde formula gives

$$\dim \mathrm{H}^0\left(\mathrm{Bun}_{\mathrm{SL}_n}, \mathcal{O}(1)\right) = n^g.$$

We have seen that classical space of conformal blocks

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Question: What happens if we modify the representation theoretical data with a more general one?

The space of generalized conformal blocks

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In a joint project with A. Gibney and N. Tarasca we are exploring how to extend these results on $\overline{M}_{g,n}$.

Going back to the classical conformal blocks

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Question: What happens if we modify the geometric input $(C, \underline{P})$ and consequently also the group G?

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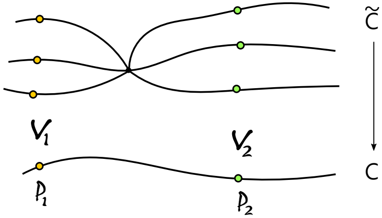

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it is possible to define the space of twisted conformal blocks

$$\mathbb{V}_\ell((\widetilde{C} \to C, \underline{P}), (\mathcal{G}, \underline{\mathcal{V}}))$$

which defines a vector bundle on the moduli space parametrizing Galois coverings of curves and satisfies properties analogous to the classical conformal blocks.

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Thank you!